Higgs bundles and related topics.

Plan: Part I: Basics of Higgs bundles
geometry of moduli space
NAH
Higher Teichmüller theory
parabolic Higgs bundles 6

Part II: topics. 6.

Today: Explain Betti, de Rham, Dolbeault moduli spaces.
Take a close look at rk 1 case.

W. Goldman and E.Z. Xia,
"Rank one Higgs bundles and representations of fundamental gps of R.S"

§0. Equivalence of deformation theories.

Defn. A deformation theory (or transformation groupoid) (S, G) consisting of a category $\mathcal{C}$ defined by a gp action as follows:

Let $\alpha: G \times S \to S$ left action.

(S, G) consists of the category $\mathcal{C}$ with $\mathrm{Obj}(\mathcal{C}) = S$ with morphism $x \xrightarrow{g} y$ corresponding to the triple $(g, x, y) \in G \times S \times S$ s.t $a(g,x) = y$.

- $e \in G$ determines the identity morphism $x \xrightarrow{e} x$.
- $x \xrightarrow{g} y$ has an inverse $y \xrightarrow{g^{-1}} x$
- composition.

<u>Defn</u>. The moduli set corresponding to such a groupoid is the set $Iso(\mathcal{L})$ of isomorphism classes of objects.

<u>Defn</u>. An equivalence of categories is a functor $F: \mathcal{A} \to \mathcal{B}$ s.t $\exists H: \mathcal{B} \to \mathcal{A}$ and $F \circ H \cong I_{\mathcal{B}}$

$H \circ F \cong I_{\mathcal{A}}$.

$\rightsquigarrow$ a bijection : $Isom(\mathcal{A}) \to Isom(\mathcal{B})$.

<u>Prop</u>. (Criterion) A functor $F: \mathcal{A} \to \mathcal{B}$ is an equiv iff (1) surjective on Isomorphism classes.

(2) Full : $F(x,y) : Mor(x,y) \to Mor(F(x), F(y))$ is surjective.

(3) Faithful : injective.

<u>§1.</u> $\boxed{\text{The Betti groupoid.}}$

Fix G a structure grp, e.g. $GL(n, \mathbb{C})$, $SL(n,\mathbb{C})$, $U(n)$;

Σ a compact smooth oriented surface with fundamental grp π.

- The objects are representations : $\pi \to G$

$$S = Hom(\pi, G)$$

- The morphisms are from G by conjugation.

$$G \times Hom(\pi, G) \longrightarrow Hom(\pi, G)$$
$$g \cdot \rho \longmapsto g^{-1} \rho g$$

Defn. The Betti groupoid is $(\mathrm{Hom}(\pi, G), G)$.

- π admits a presentation
$$\langle A_1, B_1, \dots, A_g, B_g \mid [A_1, B_1] \cdots [A_g, B_g] = 1 \rangle$$

The map
$$\mathrm{Hom}(\pi, G) \hookrightarrow G^{2g}$$
$$\rho \longmapsto (\rho(A_1), \rho(B_1), \dots, \rho(A_g), \rho(B_g)).$$

embeds $\mathrm{Hom}(\pi, G)$ as a Zariski-closed subset of G^{2g} defined $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1$. (*);

- If G is abelian, it acts trivially on $\mathrm{Hom}(\pi, G)$. The condition (*) is automatically satisfied.

So $\mathrm{Hom}(\pi, G)/G \cong \mathrm{Hom}(\pi, G) \cong G^{2g}$.
$$\mathrm{Hom}(\pi, G)/G = \mathrm{Isom}((\mathrm{Hom}(\pi, G), G)).$$

Will apply this to $G = \mathbb{C}^*, U(1), \mathbb{R}^+$.

§2. $\boxed{\text{The de Rham groupoid}}$

Let E be a smooth complex vector bundle over Σ.

$\mathcal{A}^k(\Sigma)$ denote the space of k-forms on Σ

$\mathcal{A}^k(\Sigma, E)$ . . . . E-valued k-forms.

Defn. A gauge transformation of E is a smooth bundle automorphism
$$\begin{array}{rcl} \xi : E & \longrightarrow & E \\ \downarrow & \circlearrowleft & \downarrow \\ \mathrm{id} : \Sigma & \longrightarrow & \Sigma \end{array}.$$

Denote by $\mathcal{G}(E)$ the group of gauge transformations of E.

Defn. (Connection)

A connection on E is an operator

$$D : \mathcal{A}^0(\Sigma; E) \longrightarrow \mathcal{A}^1(\Sigma, E)$$

s.t. $\quad D(fs) = fD(s) + df \wedge Ds.$

Such a map extends to $D: \mathcal{A}^p(\Sigma; E) \to \mathcal{A}^{p+1}(\Sigma; E)$.

Denote by $\mathcal{U}(E)$ the space of all connections on E.

Note that fix a connection D_0, an arbitrary connection

$$D = D_0 + \eta \quad \text{for} \quad \eta \in \mathcal{A}^1(\Sigma; \mathrm{End}(E)).$$

So $\mathcal{U}(E)$ is an affine space modeled on $\mathcal{A}^1(\Sigma; \mathrm{End}(E))$.

Defn. (Curvature)

The curvature of a connection D is

defined as $\quad F(D)s = D \circ D(s)$,

turns out to be an $\mathrm{End}(E)$-valued 2-form

$$F(D) \in \mathcal{A}^2(\Sigma; \mathrm{End}(E)).$$

Call D flat if $F(D) = 0$.

Denote by $\mathcal{F}(E)$ the space of flat connections on E.

(Note that for the existence of a flat connection, require $\deg(E) = 0$.)

- <u>The gauge action on connections</u>

$\xi^* D$ is defined as $\quad (\xi^* D)(s) = D(\xi . s)$

for $\xi \in \mathcal{G}(E)$.

$$\xi\cdot D := (\xi^{-1})^* D.$$

Locally, w.r.t a frame e,

$$D = d + \eta \quad (\text{i.e } De = e\eta)$$

Then $\xi^* D = d + g^{-1}\eta g + g^{-1}dg$

(Here, g is the local expression of ξ w.r.t e.)

i.e. $\xi e = eg$.

$$(\xi^* D)(e) = D(eg) = e(\eta g + dg)$$
$$= eg(g^{-1}\eta g + g^{-1}dg).$$

- $F(\xi^* D) = \xi^*(F(D))$.

Hence, $\mathcal{G}(E)$ preserves flatness.

<u>Defn</u>. The de Rham groupoid is $(\mathcal{F}(E), \mathcal{G}(E))$.

<u>§3.</u> Equivalence between Betti and de Rham groupoids

Start from a flat connection D on a vector bundle E, want to obtain a rep $\rho: \pi \to GL(n, \mathbb{C})$.

Locally, w.r.t a frame e, $De = e\cdot\eta$.

Over a smooth path $\gamma: [0,1] \to \Sigma$, parallel transport defines a linear map between the fibers $P_{\gamma(t)}: E_{\gamma(0)} \longrightarrow E_{\gamma(t)}$.

That is, $P_{\gamma(t)}(v)$ is parallel w.r.t D, for $v \in E_{\gamma(0)}$.

Suppose $v = (e \circ \gamma(0)) \cdot f(0) \in E_{\gamma(0)}$.

Then $P_{\gamma(t)}(v) = (e \circ \gamma(t)) \cdot \underline{g(t)} \cdot f(0)$ is parallel to D

$$\Leftrightarrow \quad D_{\partial/\partial t}\big((e \circ \gamma(t)) \cdot g(t) \cdot f(0) \big) = 0$$

$$\Leftrightarrow \quad (e \circ \gamma(t)) \big(\underbrace{\eta \circ \gamma(t) \cdot g(t) + dg(t)}_{} \big)(\partial/\partial t) \cdot f(0) = 0$$

$$\Leftrightarrow \quad g'(t) + (\eta \circ \gamma(t)) \cdot g(t) = 0$$

$$\Leftrightarrow \quad g(t) = \exp\left(- \int_0^t \gamma^* \eta \right)$$

<u>Fact</u>: Flatness of D implies the parallel transport only depends on homotopic class of γ relative to its endpts.

Now we obtain a homomorphism: fix a pt $p \in E_{x_0}$.

$$\mathrm{hol}_p(D) : \pi_1(X, x_0) \longrightarrow GL(n, \mathbb{C})$$

$$\gamma \longmapsto \big(P_\gamma : E_{x_0} \to E_{x_0} \big)^{-1}$$

with $x_0 = \gamma(0)$ and $x_0 = \gamma(1)$,

w.r.t a fixed frame e at E_{x_0}.

<u>Thm</u>: The holonomy functor

$$\mathrm{hol}_p : (\mathcal{F}(E), G(E)) \longrightarrow (\mathrm{Hom}(\pi, GL(n,\mathbb{C})), GL(n,\mathbb{C}))$$

is an equivalence of groupoids.

Pf: • surjective on isomorphism classes.

Given a rep $\rho \in \mathrm{Hom}(\pi, GL(n,\mathbb{C}))$, we construct a flat vector bundle $\mathbb{C}_\rho \to \Sigma$ as follows:

the grp π acts on the total space $\check{\Sigma} \times \mathbb{C}^n$ by

$$\gamma \cdot (\check{s}, x) := (\underset{\substack{\uparrow \\ \text{deck transformation}}}{\overset{\text{“}\check{s}\cdot\gamma^{-1}\text{”}}{\gamma \cdot \check{s}}}, \underline{\rho(\gamma) x}) \quad \forall \gamma \in \pi.$$

The quotient $(\check{\Sigma} \times \mathbb{C}^n)/\pi$ is the total space of a smooth vector bundle $\mathbb{C}_\rho \xrightarrow{p} \Sigma$, which carries a natural flat connection D as the descending of $D_0 = d$ on $\check{\Sigma} \times \mathbb{C}^n$.

$[[\check{\Sigma}, \underset{\substack{\uparrow \\ \text{constant}}}{v}]]$ is parallel to D.

So this D gives holonomy ρ up to conjugation ∎

– Full and faithful (need to check)

§4. $\boxed{\text{Rank 1 case for equivalence between Betti and de Rham moduli spaces.}}$

Let E be a trivial complex line bundle over Σ.

A trivialization τ is a global frame of E.

- The gauge transformation $\xi \in \mathcal{G}(E)$ is determined by a smooth map $g : \Sigma \to \mathbb{C}^*$ via

$$\xi(\tau) = g \cdot \tau.$$

$$\mathcal{G}(E) \cong \mathrm{Map}(\Sigma, \mathbb{C}^*).$$

The subgrp $\mathcal{G}_u(E) \cong \mathrm{Map}(\Sigma, U(1))$

Let $\mathrm{Map}(\Sigma, \mathbb{C}^*)^\circ$ denote the component containing the constant map.

$\mathcal{G}(E)^\circ =$

$$G(E)/G(E)^{\circ} = \pi_0(G(E))$$

Note that $\quad Map(\Sigma, \mathbb{C}^*)^{\circ} \cong \mathcal{A}^0(\Sigma)$

$$g \longmapsto \log g .$$

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ \Sigma \arrow[r, "g"'] \arrow[ur, dashed, "\log g"] & \mathbb{C}^* \end{tikzcd} \quad \text{iff} \quad g_*: \pi_1\Sigma \to \pi_1(\mathbb{C}^*) \text{ is trivial}$$

So $\quad \mathcal{F}(E)/G(E) = \left(\mathcal{F}(E)/G(E)^{\circ}\right) / \pi_0(G(E)).$

- On E, there is a unique connection D_0 s.t. $D_0\tau = 0$.

  Any connection D is of the form

  $$D = D_0 + \eta, \quad \eta \in \mathcal{A}^1(\Sigma).$$

  D is flat $\Longleftrightarrow d\eta = 0$.

  $\xi^*(D_0+\eta) = D_0 + \eta + g^{-1}dg.$ $(\xi \leftrightarrow g \in Map(\Sigma, \mathbb{C}^*))$

  If $g \in Map(\Sigma, \mathbb{C}^*)^{\circ}$, $g^{-1}dg = d\log g$.

  So $\quad \mathcal{F}(E)/G(E)^{\circ} \cong Z^1(\Sigma)/B^1(\Sigma) = H^1(\Sigma).$

The Betti moduli space is $Hom(\pi, \mathbb{C}^*) \cong Hom(\pi, S^1) \times Hom(\pi, \mathbb{R}^+)$

The de Rham moduli space:

- $$\begin{array}{ccccc} \mathcal{F}(E) & = & \mathcal{F}_u(E) & \times & \mathcal{A}^1(\Sigma, \mathbb{R}) \\ D_0+\eta & & D_0 + i\,\mathrm{Im}\,\eta & & \mathrm{Re}\,\eta \end{array}$$

- $\mathcal{G}(E) \cong \mathrm{Map}(\Sigma, \mathbb{C}^*) = \underset{\mathcal{G}_u(E)}{\overset{\shortparallel}{\mathrm{Map}(\Sigma, S^1)}} \times \mathrm{Map}(\Sigma, \mathbb{R}^+)$

$$g \mapsto (g_u, g_r)$$

$$g^*(D_0+\eta) = D_0 + \eta + g^{-1}dg$$
$$= \left(D_0 + i\,\mathrm{Im}\,\eta + \underline{(g_u^{-1} d g_u)}\right) + \Big(\underbrace{g_r^{-1} d g_r}_{\overset{\shortparallel}{d\log g_r}} + \mathrm{Re}\,\eta\Big)$$

- $\mathcal{F}(E)/\mathcal{G}(E) \cong \mathcal{F}_u(E)/\mathcal{G}_u(E) \times H^1(\Sigma, \mathbb{R})$.

$$\cong \Big(\underbrace{\mathcal{F}_u(E)/\mathcal{G}_u^0(E)}_{H^1(\Sigma, i\mathbb{R})}\Big)\Big/\pi_0(\mathcal{G}_u(E)) \times H^1(\Sigma, \mathbb{R})$$

(will see $\cong H^1(\Sigma, i\mathbb{R})/H^1(\Sigma, \mathbb{Z}) \times H^1(\Sigma, \mathbb{R})$.)

The equivalence between the moduli spaces is given by

$$\mathrm{hol}_p : \mathcal{F}(E) \longrightarrow \mathrm{Hom}(\pi, \mathbb{C}^*)$$
$$D_0 + \eta \longmapsto \left(\gamma \mapsto \exp\left(\int_\gamma \eta\right)\right)$$

Restrict to $\mathrm{hol}_p : \mathcal{F}_u(E) \longrightarrow \mathrm{Hom}(\pi, U(1))$

<u>Claim:</u> $\ker(\mathrm{hol}_p) = \mathcal{G}(E)$.

Assuming the claim: hol_p desends to a map

$$\rho : \Big(\underbrace{\mathcal{F}_u(E)/\mathcal{G}_u(E)}_{\overset{\shortparallel}{H^1(X, i\mathbb{R})}}\Big)\Big/\pi_0(\mathcal{G}_u(E)) \longrightarrow \mathrm{Hom}(\pi, U(1)) = \mathrm{Hom}(\pi, i\mathbb{R})\big/\mathrm{Hom}(\pi, 2\pi i\mathbb{Z})$$

$$\Rightarrow \pi_0(\mathcal{G}_\mathbb{C}(E)) \cong H^1(\Sigma, \mathbb{Z}).$$

$$\| $$

$$\mathbb{Z}\langle w_1, \dots, w_{2g}\rangle$$

where $w_1, \dots, w_{2g}$ has period $\in 2\pi i\mathbb{Z}$ and form a basis.

<u>Pf of Claim:</u>

"$\subset$" If $\eta \in H^1(\Sigma, i\mathbb{R})$ s.t $\exp(\int_\gamma \eta) = 1 \quad \forall \gamma \in \pi$.

define $g(p) = \exp \int_{x_0}^{p} \eta$ well-defined.

as a fcn $g: \Sigma \to \mathbb{C}^*$.

and $\eta = g^{-1}dg$.

"$\supset$" Given $g^{-1}dg$ for $g: \Sigma \to \mathbb{C}^*$,

one can lift g to $\check{g}: \check{\Sigma} \to \mathbb{C}^*$.

then $\check{g}^{-1}d\check{g}$ is exact on $\check{\Sigma}$.

$$\check{g}^{-1}d\check{g} = d\log\check{g}.$$

Take a lift $\check{\gamma}$ of γ to $\check{\Sigma}$,

then $\int_\gamma g^{-1}dg = \int_{\check{\gamma}(0)}^{\check{\gamma}(1)} \check{g}^{-1}d\check{g} = \int_{\check{\gamma}(0)}^{\check{\gamma}(1)} d\log\check{g}$

$= \log\check{g}(\check{\gamma}(1)) - \log\check{g}(\check{\gamma}(0))$

$\in 2\pi i\mathbb{Z}$.

$\Rightarrow \exp \int_\gamma g^{-1}dg = 1.$ $\blacksquare$