§5. Quantization and Obstruction

<u>Recall</u>: QFT $\rightsquigarrow (\mathcal{E}, Q, \omega)$ ∞-dim (-1)-symplectic

Classical data:

$$\begin{array}{ccc} \mathcal{O}_{loc}(\mathcal{E}) & \subset & \mathcal{O}(\mathcal{E}) \\ \cup & & \\ I_0 = \int_X \mathcal{L} & & \text{classical interaction} \end{array}$$

which solves
$$Q I_0 + \frac{1}{2}\{I_0, I_0\}_0 = 0 \qquad \text{(CME)}$$

$\{-,-\}_0$ BV bracket w.r.t. BV kernel $K_0 = \omega^{-1}$

- $\{-,-\}_0$ well-defined on $\mathcal{O}_{loc}(\mathcal{E})$
- Δ_0 ill-defined on $\mathcal{O}_{loc}(\mathcal{E})$, $\mathcal{O}(\mathcal{E})$

$$\Rightarrow \quad \cancel{Q I_0 + \hbar \Delta_0 I_0 + \frac{1}{2}\{I_0, I_0\} = 0}$$

this naive QME doesn't make sense.

One way-out: use regularization to define effective QME

$$K_0 = K_r + Q(P_r)$$

$$\Delta_r \text{ well-defined on } \mathcal{O}(\mathcal{E})$$

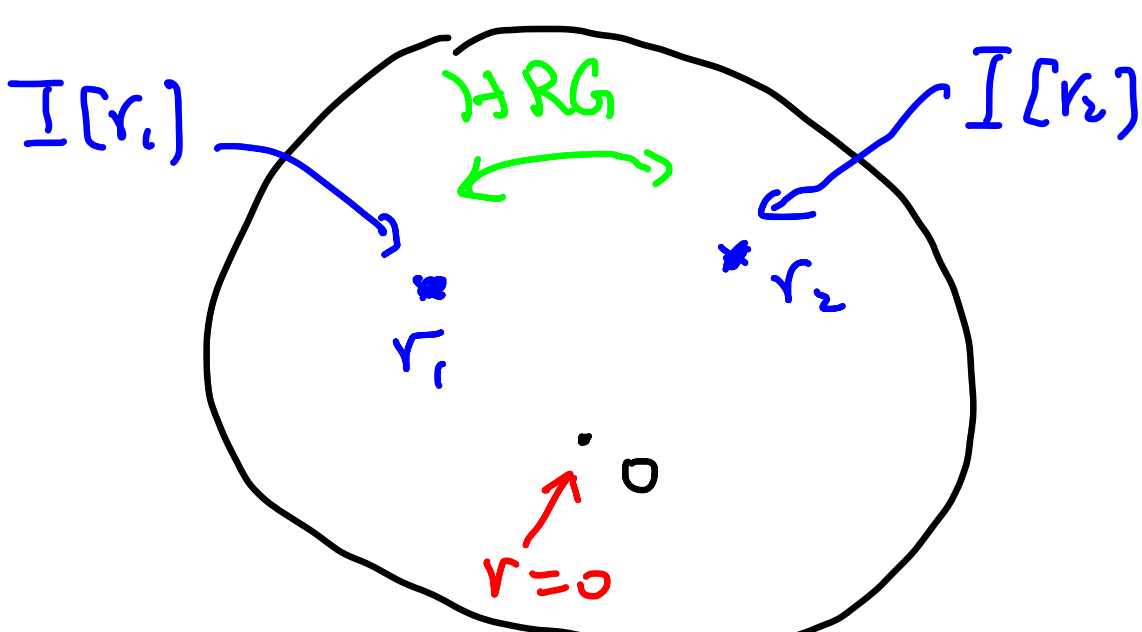
For each r, we construct $I[r] \in \mathcal{O}(\mathcal{E})$ solving

$$(Q + \hbar \Delta_r) e^{I[r]/\hbar} = 0 \quad \text{effective QME}$$

and different regularizations are related by

$$e^{I[r']/\hbar} = e^{\hbar \partial_{P_r^{r'}}} e^{I[r]/\hbar} \quad \text{homotopy RG}$$

Effective QME and Homotopy RG are compatible.



- Heat Kernel regularization

Typically, fix a choice of metric, we have

- $Q^+ : \mathcal{E} \to \mathcal{E}$ the adjoint of $Q : \mathcal{E} \to \mathcal{E}$

- $[Q, Q^+] = QQ^+ + Q^+Q$ generalized Laplacian

$\Longrightarrow$ Heat operator $e^{-L[Q,Q^+]}$ for $L > 0$

and let $K_L \in \mathrm{Sym}^2(\mathcal{E})$ be its kernel by

$$\left(e^{-L[Q,Q^+]}\alpha\right)(x) = \int dy \left\langle K_L(x,y), \alpha(y)\right\rangle$$

for $\alpha \in \mathcal{E}$

Here $\langle -,- \rangle$ is the pairing from ω

$$\begin{array}{ccc} K_L & & \alpha \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathcal{E} \otimes \mathcal{E} & & \mathcal{E} \end{array}$$

pair this two factors

- $K_0 = \lim_{L\to 0} K_L$ is the δ-function distribution ω^{-1}

- $K_L \in \mathrm{Sym}^2(\mathcal{E})$ and smooth for $L>0$.

Let P_L be the kernel of the operator

$$\int_0^L Q^+ e^{-t[Q,Q^+]}\,dt$$

Explicitly, we have

$$P_L = \int_0^L (Q^+\otimes 1)\, K_t\, dt$$

The operator equation

$$\left[Q, \int_0^L Q^+ e^{-t[Q,Q^+]}\,dt\right]$$

$$= \int_0^L [Q,Q^+]\, e^{-t[Q,Q^+]}\,dt$$

$$= 1 - e^{-L[Q,Q^+]}$$

This is translated to the kernel equation

$$K_0 - K_L = (Q\otimes 1 + 1\otimes Q) P_L$$

Or simply written as

$$K_0 - K_L = Q(P_L)$$

We can use K_L to define effective QME

For $0 < \varepsilon < L$, similarly the operator equation

$$\left[Q, \int_\varepsilon^L Q^+ e^{-t[Q,Q^+]}\right] = e^{-\varepsilon[Q,Q^+]} - e^{-L[Q,Q^+]}$$

$$\Downarrow$$

$$K_\varepsilon - K_L = (Q\otimes 1 + 1\otimes Q) P_\varepsilon^L$$

where $P_\varepsilon^L = \int_\varepsilon^L (Q^+\otimes 1) K_t \, dt$ "Regularized propagator"

We can use P_ε^L to connect

$$\text{Effective QME at } \varepsilon \xrightarrow{e^{\hbar \partial_{P_\varepsilon^L}}} \text{Effective QME at } L$$

- Constructing effective BV quantization

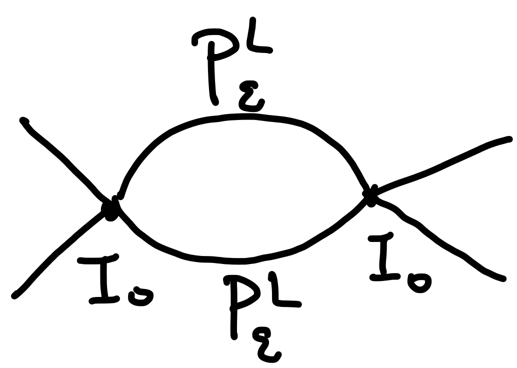
(using Heat Kernel regularization)

Step 1 : The method of counter-term

Let $I_0 \in \mathcal{O}_{loc}(\mathcal{E})$ be the classical interaction

Since P_0^L is singular, the limit

$$\lim_{\varepsilon \to 0} e^{\hbar \partial_{P_\varepsilon^L}} e^{I_0/\hbar} \quad \text{doesn't exist}$$



divergent as $\varepsilon \to 0$ for loop diagrams

Can find $I^\epsilon \in \hbar\, \mathcal{O}_{loc}(\mathcal{E})[[\hbar]]$ ε-dependent singular as $\varepsilon \to 0$

such that

$$\lim_{\varepsilon \to 0} e^{\hbar \partial_{P_\varepsilon^L}} e^{(I_0 + I^\epsilon)/\hbar} \quad \text{exists}$$

$$:= e^{I[L]^{\text{Naive}}/\hbar}$$

I^ϵ : Counter-term

Such defined $I[L]^{\text{Naive}}$ has the following advantage

for $0 < L_1 < L_2$,

$$e^{I[L_2]^{\text{Naive}}/\hbar} = \lim_{\varepsilon \to 0} e^{\hbar \partial_{P^{L_2}_{\varepsilon}}} e^{(I_0 + I^{\epsilon})/\hbar}$$

using $P^{L_2}_{\varepsilon} = P^{L_2}_{L_1} + P^{L_1}_{\varepsilon}$

$$= \lim_{\varepsilon \to 0} e^{\hbar \partial_{P^{L_2}_{L_1}}} e^{\hbar \partial_{P^{L_1}_{\varepsilon}}} e^{(I_0 + I^{\epsilon})/\hbar}$$

Since $P^{L_2}_{L_1}$ is smooth

$$= e^{\hbar \partial_{P^{L_2}_{L_1}}} \lim_{\varepsilon \to 0} e^{\hbar \partial_{P^{L_1}_{\varepsilon}}} e^{(I_0 + I^{\epsilon})/\hbar}$$

$$= e^{\hbar \partial_{P^{L_2}_{L_1}}} e^{I[L_1]^{\text{Naive}}/\hbar}$$

In other words, the family

$\{ I[L]^{\text{Naive}} \}_{L > 0}$ satisfies Homotopy RG.

$$I[L_1]^{\text{Naive}} \overset{\text{HRG}}{\rightsquigarrow} I[L_2]^{\text{Naive}}$$

However, $\{I^{\text{Naive}}[L]\}$ may not satisfy QME

· **Step 2**. Adjust I^{ϵ} to $\tilde{I}^{\epsilon}$ by finding further corrections such that

$$e^{I[L]/\hbar} = \lim_{\varepsilon\to 0} e^{\hbar \partial_{P_\varepsilon}^L} e^{(I_0+\tilde{I}^{\epsilon})/\hbar}$$

the defined limit $I[L]$ satisfies QME

Rmk: · "Step 1" is always possible

· "Step 2" is NOT always possible.

It might have obstructions. In physics, this is called "gauge anomaly"

Rmk: There are cases where ε-dependent counter-terms are not required in the sense that

the limit $e^{I[L]/\hbar} = \lim_{\varepsilon\to 0} e^{\hbar\partial_{P_\varepsilon^L}} e^{I/\hbar}$ exists

for a large class of local $I \in \mathcal{O}_{loc}(\mathcal{E})[[\hbar]]$

Such theory is called "UV-finite"

Then we can explore the meaning of effective QME $(Q+\hbar\Delta_L)e^{I[L]/\hbar}=0$

or $\quad Q\,I[L] + \hbar\Delta_L I[L] + \frac{1}{2}\{I[L], I[L]\}_L = 0$

$$\Big\Downarrow \quad \text{upshot: } L\to 0 \text{ has a meaning}$$

$$QI + \frac{1}{2}[I,I] = 0$$

$\rightsquigarrow$ quantum deformed bracket

We will explain two main UV-finite examples

① Topological theory (Chern-Simons type)

In particular, we will discuss top. quantum mechanics

② 2d chiral theory

· Deformation - Obstruction theory

Let's first consider the quantization problem

$$\text{CME} \rightsquigarrow \text{QME}$$

in a DGBV $(\mathcal{A}, Q, \Delta)$.

Let $I_0 \in \mathcal{A}_0$ solve the CME

$$Q I_0 + \frac{1}{2}\{I_0, I_0\} = 0$$

Our goal is to find

$$I = I_0 + I_1 \hbar + I_2 \hbar^2 + \cdots \in \mathcal{A}_0[[\hbar]]$$

Solving the QME

$$QI + \hbar\Delta I + \frac{1}{2}\{I,I\} = 0.$$

Strategy: find $I_1, I_2, \dots$ in order of $\hbar$.

$$QI + \hbar\Delta I + \frac{1}{2}\{I,I\} = O(\hbar^{n+1}) \quad (n \geq 0)$$

- $\boxed{n=0}$: this is the initial data of CME

$$QI_0 + \frac{1}{2}\{I_0, I_0\} = 0$$

- $\boxed{n=1}$: $\hbar$-term gives

$$QI_1 + \Delta I_0 + \{I_0, I_1\} = 0$$

We need to find I_1 solving the above equation

Let us write it as

$$QI_1 + \{I_0, I_1\} = -\Delta I_0$$

For convenience, let us denote

$$\delta = Q + \{I_0, -\}$$

CME implies that $\delta^2 = 0$. We need to solve

$$\delta I_1 = -\Delta I_0$$

Key observation: $\delta(-\Delta I_0) = 0$. (Exercise!)

So we see that $-\Delta I_0$ is δ-closed.

The solvability of I_1 asks whether $-\Delta I_0$ is δ-exact

Let $\Theta_1 = \Delta I_0$ and let $[\Theta_1] \in H^1(A, \delta)$ be the corresponding δ-coh. class. Then

Prop: I_1 can be solved $\Longleftrightarrow [\Theta_1] = 0$ in $H^1(A, \delta)$

Assume $[\Theta_1] = 0$. Let I_1 and $\tilde{I}_1$ be two sol'ns.

$$\Rightarrow \quad \delta(I_1 - \tilde{I}_1) = 0$$

$$\Rightarrow \quad \tilde{I}_1 - I_1 \text{ is } \delta\text{-closed.}$$

Also, for any $J \in \mathcal{A}_0$, the solution

$$I_1 + \delta J \underset{\text{equivalent}}{\overset{\text{gauge}}{\sim}} I_1$$ in a suitable sense.

(Solving a family version of QME along an interval)

Prop. If I_1 can be solved, then

$$\{\text{sol'n of } I_1\} / \text{gauge} = H^0(\mathcal{A}, \delta)$$

• $\boxed{n > 1}$: Assume we have found

$$I_{<k} := I_0 + I_1 \hbar + \cdots + I_{k-1} \hbar^{k-1} \quad \text{solving}$$

$$Q I_{<k} + \hbar \Delta I_{<k} + \frac{1}{2}\{I_{<k}, I_{<k}\} = O(\hbar^k)$$

Let's consider the problem of solving I_k.

The above equation can be written as

$$(Q+\hbar\Delta)\, e^{I_{<k}/\hbar} = O(\hbar^{k-1})\, e^{I_{<k}/\hbar}$$

We want to find I_k s.t.

$$(Q+\hbar\Delta)\, e^{(I_{<k}+I_k\hbar^k)/\hbar} = O(\hbar^k)\, e^{(I_{<k}+I_k\hbar^k)/\hbar}$$

Let us write

$$(Q+\hbar\Delta)\, e^{I_{<k}/\hbar} = \left(\underset{\uparrow}{\Theta_k}\hbar^{k-1} + O(\hbar^k)\right) e^{I_{<k}/\hbar}$$

the leading term in $O(\hbar^{k-1})$

Explicitly, we have

$$Q\, I_{<k} + \hbar\Delta I_{<k} + \frac{1}{2}\{I_{<k}, I_{<k}\} = \Theta_k \hbar^k + O(\hbar^{k+1})$$

Similarly, we need to solve

$$Q(I_{<k}+\hbar^k I_k) + \hbar(\Delta I_{<k} + \hbar^k I_k) + \frac{1}{2}\{I_{<k}+\hbar^k I_k, I_{<k}+\hbar^k I_k\} = O(\hbar^{k+1})$$

This is equivalent to

$$Q I_k + \{I_0, I_k\} = \Theta_k$$

or $$\boxed{\delta I_k = \Theta_k}$$

<u>Claim</u>: Θ_k is δ-closed.

<u>Sketch</u>: Since $(Q+\hbar\Delta)$ to both sides of

$$(Q+\hbar\Delta)e^{I_{<k}/\hbar} = \left(\Theta_k \hbar^{k-1} + O(\hbar^k)\right)e^{I_{<k}/\hbar}$$

and use the fact that $(Q+\hbar\Delta)^2 = 0$

$$\Rightarrow \quad Q\,\Theta_k + \{I_0, \Theta_k\} = 0 \quad \#$$

The solvability of I_k asks

whether Θ_k is δ-exact

Prop: Solvability of $I_k \Longleftrightarrow [\mathcal{O}_k]=0$ in $H^1(\mathcal{A},\delta)$

Moreover, if I_k can be solved, then

$$\{\text{Sol'n of } I_k\}/_{\text{gauge}} = H^0(\mathcal{A},\delta)$$

Rmk:
- $[\mathcal{O}_k] \in H^1(\mathcal{A},\delta)$ is the obstruction class (gauge anomaly) for solving QME up to $\hbar^k$
- $H^1(\mathcal{A},\delta)$: obstruction space
- $H^0(\mathcal{A},\delta)$: tangent space (deformation space)

In particular, we have proved

Theorem: If $H^1(\mathcal{A}, Q+\{I_0,-\})=0$, then there exits a quantization $I = I_0 + \hbar I_1 + \cdots$ solving the QME $QI + \hbar\Delta I + \frac{1}{2}\{I,I\}=0$.

· Back to effective QME

In the QFT case, things are more complicated since we need to deal w/. regularization. However, the good thing is that the analogue of deformation-obstruction theory still exists. The relevant complex is

$$\left(\mathcal{O}_{loc}(\mathcal{E}),\ Q+\{I_0,-\} \right)$$

Note that $Q+\{I_0,-\}$ is well-defined on local functionals, and CME implies that $\left(\mathcal{O}_{loc}(\mathcal{E}), Q+\{I_0,-\}\right)$ indeed forms a complex.

Similar to the above discussion, we have

Theorem. The obstruction space for effective BV quantization of I_0 is given by

$$H^1(\mathcal{O}_{loc}(\mathcal{E}), Q+\{I_0,-\})$$

The tangent space (deformation space) is

$$H^0(\mathcal{O}_{loc}(\mathcal{E}), Q+\{I_0,-\})$$

Rmk. The locality is important and allows the computation of the above cohomologies via D-module methods.

Ref Today: The above theorem has many different set-up and versions in the literature. The discussion we follow here is

Costello: Renormalization and effective field theory

where you can also find references for classical works in the literature.