

Higgs bundles and related topics.

Lecture

Plan: Part I: Basics of Higgs bundles
geometry of moduli space

6

NAH

Higher Teichmüller theory
parabolic Higgs bundles

Part II: topics.

6.

Explain Betti, de Rham, Dolbeault
moduli spaces.

Take a close look at rk 1 case.

W. Goldman and E.Z. Xia,

"Rank one Higgs bundles and representations of
fundamental groups of R.S"

§0. Equivalence of deformation theories.

Defn. A deformation theory (or transformation groupoid)
(S, G) consisting of a category \mathcal{C} defined by
a grp action as follows:

Let $\alpha: G \times S \rightarrow S$ left action.

(S, G) consists of the category \mathcal{C} with $Obj(\mathcal{C}) = S$
with morphism $x \xrightarrow{g} y$ corresponding to

- the triple $(y, x, y) \in G \times S \times S$ s.t $\alpha(y, x) = y$.
- $e \in G$ determines the identity morphism $x \xrightarrow{e} x$.
- $x \xrightarrow{g} y$ has an inverse $y \xrightarrow{g^{-1}} x$
- composition.

Defn. The moduli set corresponding to such a groupoid
is the set $\text{Iso}(\mathcal{L})$ of isomorphism classes of objects.

Defn. An equivalence of categories is a functor $F: \mathcal{A} \rightarrow \mathcal{B}$
st $\exists H: \mathcal{B} \rightarrow \mathcal{A}$ and $F \circ H \cong I_{\mathcal{B}}$
 $H \circ F \cong I_{\mathcal{A}}$.

→ a bijection: $\text{Isom}(\mathcal{A}) \rightarrow \text{Isom}(\mathcal{B})$.

Prop. (Criterion) A functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is an equiv
iff (1) subjective on Isomorphism classes.
(2) Full: $F(x, y): \text{Mor}(x, y) \rightarrow \text{Mor}(F(x), F(y))$
is surjective.
(3) Faithful: injective.

§1. The Betti groupoid.

Fix G a structure grp, e.g. $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $U(n)$,
 Σ a compact smooth oriented surface with
fundamental grp π .

- The objects are representations: $\pi \rightarrow G$
 $S = \text{Hom}(\pi, G)$
- The morphisms are from G by conjugation.
 $G \times \text{Hom}(\pi, G) \rightarrow \text{Hom}(\pi, G)$
 $g \cdot \rho \mapsto g^{-1}\rho g$

Defn. The Betti groupoid is $(\text{Hom}(\pi, G), G)$.

- π admits a presentation

$$\langle A_1, B_1, \dots, A_g, B_g \mid [A_1, B_1] \dots [A_g, B_g] = 1 \rangle$$

The map $\text{Hom}(\pi, G) \hookrightarrow G^{2g}$
 $p \mapsto (p(A_1), p(B_1), \dots, p(A_g), p(B_g))$.

embeds $\text{Hom}(\pi, G)$ as a Zariski-closed
subset of G^{2g} defined $[A_1, B_1] \dots [A_g, B_g] = 1$. $(*)$

- If G is abelian, it acts trivially on $\text{Hom}(\pi, G)$.

The condition $(*)$ is automatically satisfied.

$$\text{So } \text{Hom}(\pi, G)/G \cong \text{Hom}(\pi, G) \cong G^{2g}.$$

$$\text{Isom}^{\text{''}}((\text{Hom}(\pi, G), G)).$$

will apply this to $G = \mathbb{C}^*, U(1), \text{IR}^+$.

§2. The de Rham groupoid

Let E be a smooth complex vector bundle over Σ .

$\mathcal{A}^k(\Sigma)$ denote the space of k -forms on Σ

$\mathcal{A}^k(\Sigma, E)$ E -valued k -forms.

Defn. A gauge transformation of E is a smooth
bundle automorphism $\beta: E \rightarrow E$

$$\downarrow \curvearrowright \downarrow$$

$$\text{id}: \Sigma \longrightarrow \Sigma$$

Denote by $G(E)$ the group of gauge transformations of E .

Defn. (Connection)

A connection on E is an operator

$$D: \mathcal{A}^0(\Sigma; E) \rightarrow \mathcal{A}^1(\Sigma; E)$$

$$\text{s.t. } D(fs) = fD(s) + df \wedge Ds.$$

Such a map extends to $D: \mathcal{A}^P(\Sigma; E) \rightarrow \mathcal{A}^{PH}(\Sigma; E)$.

Denote by $\mathcal{U}(E)$ the space of all connections on E .

Note that fix a connection D_0 , an arbitrary connection

$$D = D_0 + \eta \quad \text{for } \eta \in \mathcal{A}^1(\Sigma; \text{End}(E)).$$

So $\mathcal{U}(E)$ is an affine space modeled on $\mathcal{A}^1(\Sigma; \text{End}(E))$.

Defn. (Curvature)

The curvature of a connection D is

$$\text{defined as } F(D)S = D_0 D(S),$$

turns out to be an $\text{End}(E)$ -valued 2-form

$$F(D) \in \mathcal{A}^2(\Sigma; \text{End}(E)).$$

Call D flat if $F(D) = 0$.

Denote by $\mathcal{F}(E)$ the space of flat connections on E .

(Note that for the existence of a flat connection,
require $\deg(E) = 0$.)

• The gauge action on connections

$$\xi^*D \text{ is defined as } (\xi^*D)(s) = \overset{\dashv}{\xi} D(s, s)$$

for $\xi \in G(E)$.

$$\xi \cdot D := (\xi^{-1})^* D.$$

Locally, w.r.t a frame e ,

$$D = d + \eta \quad (\text{i.e. } De = e\eta)$$

$$\text{Then } \xi^* D = d + g^{-1} \eta (g + g^{-1} dg)$$

(Here, g is the local expression of ξ w.r.t e .)

$$\text{i.e. } \xi e = eg.$$

$$\begin{aligned} (\xi^* D)(e) &= D(eg) = e(\eta g + dg) \\ &= eg(g^{-1}\eta g + g^{-1}dg). \end{aligned}$$

- $F(\xi^* D) = \xi^*(F(D))$.

Hence, $G(E)$ preserves flatness.

Defn. The de Rham groupoid is $(\mathcal{F}(E), G(E))$.

§3. Equivalence between Betti and de Rham groupoids

Start from a flat connection D on a vector bundle E , want to obtain a rep $p: \pi \rightarrow GL(n, \mathbb{C})$.

Locally, w.r.t a frame e , $De = e \cdot \eta$.

Over a smooth path $\sigma: [0, 1] \rightarrow \Sigma$,

parallel transport defines a linear map between the fibers $P_{\sigma(t)}: E_{\sigma(0)} \rightarrow E_{\sigma(t)}$.

That is, $P_{\sigma(t)}(v)$ is parallel w.r.t D , for $v \in E_{\sigma(0)}$.

Suppose $v = (e \circ \gamma(0)) \cdot f(0) \in E_{\gamma(0)}$.

Then $P_{f(t)}(v) = (e \circ \gamma(t)) \cdot \underline{g(t)} \cdot f(0)$ is parallel to D

$$\begin{aligned} &\Leftrightarrow D_{\gamma(t)}((e \circ \gamma(t)) \cdot g(t) \cdot f(0)) = 0 \\ &\Leftrightarrow (e \circ \gamma(t)) \left(\eta \circ \gamma(t) \cdot g(t) + dg(t) \cdot f(0) \right)^{(\partial/\partial t)} = 0 \\ &\Leftrightarrow g'(t) + (\eta \circ \gamma(t)) \cdot g(t) = 0 \\ &\Leftrightarrow g(t) = \exp(-\int_0^t \gamma^* \eta) \end{aligned}$$

Fact: Flatness of D implies the parallel transport only depends on homotopic class of γ relative to its endpoints.

Now we obtain a homomorphism: fix a pt $p \in E_{x_0}$.

$$\text{hol}_p(D) : \pi_1(X, x_0) \rightarrow GL(n, \mathbb{C})$$

$$\gamma \mapsto (P_\gamma : E_{x_0} \xrightarrow{\sim} E_{\gamma(1)})^{-1}$$

w.r.t a fixed frame e at E_{x_0} .

Thm: The holonomy functor

$$\text{hol}_p : (\mathcal{T}(E), G(E)) \rightarrow (\text{Hom}(\pi_1, GL(n, \mathbb{C})), GL(n, \mathbb{C}))$$

is an equivalence of groupoids.

Pf: • surjective on isomorphism classes.

Given a rep $p \in \text{Hom}(\pi_1, GL(n, \mathbb{C}))$, we construct a flat vector bundle $\mathbb{C}_p \rightarrow \Sigma$ as follows:

the grp π acts on the total space $\Sigma \times \mathbb{C}^n$ by
 $f \cdot (\xi, x) := (\xi \cdot \tilde{\xi}, \underbrace{P(\alpha)x}_{\text{deck transformation}}) \quad \forall \alpha \in \pi.$

The quotient $(\Sigma \times \mathbb{C}^n)/\pi$ is the total space of
a smooth vector bundle $\mathbb{C}_p \xrightarrow{P} \Sigma$,
which carries a natural flat connection D as
the descending of $D_0 = d$ on $\Sigma \times \mathbb{C}^n$.

$[\Sigma, \mathbb{C}]$ is parallel to D .
 \uparrow constant

So this D gives holonomy P up to conjugation. \square

- Full and faithful (need to check)

§4. Rank 1 case for equivalence between Betti
and de Rham moduli spaces.

Let E be a trivial complex line bundle over Σ .

A trivialization τ is a global frame of E .

- The gauge transformation $\xi \in G(E)$ is determined by
a smooth map $g: \Sigma \rightarrow \mathbb{C}^*$ via
 $\hat{\xi}(\tau) = g \cdot \tau.$

$$G(E) \cong \text{Map}(\Sigma, \mathbb{C}^*).$$

The subgroup $G_{\text{U}}(E) \cong \text{Map}(\Sigma, U(1))$

Let $G(E)^0 = \text{Map}(\Sigma, \mathbb{C}^*)^\circ$ denote the component containing
the constant map.

$$G(E)/_{G(E)^\circ} = \text{Tor}(G(E))$$

Note that $\text{Map}(\Sigma, \mathbb{C}^*)^\circ \cong A^0(\Sigma)$

$$g \mapsto \log g.$$

$$\begin{array}{ccc} \log_g: & \mathbb{C} & \downarrow \exp \\ \Sigma \xrightarrow{g} & \mathbb{C}^* & \text{iff } g_*: \pi_1 \Sigma \rightarrow \pi_1(\mathbb{C}^*) \text{ is trivial} \end{array}$$

$$\text{So } \mathfrak{f}(E)/_{G(E)} = \left(\mathfrak{f}(E)/_{G(E)^\circ} \right) / \underline{\text{Tor}(G(E))}.$$

- On E , there is a unique connection D_0 s.t. $D_0 \tau = 0$.

Any connection D is of the form

$$D = D_0 + \eta, \quad \eta \in A^1(\Sigma).$$

D is flat $\Leftrightarrow d\eta = 0$.

$$\xi^*(D_0 + \eta) = D_0 + \eta + g^* dg. \quad (\xi \leftrightarrow g \in \text{Map}(\Sigma, \mathbb{C}^*))$$

If $g \in \text{Map}(\Sigma, \mathbb{C}^*)^\circ$, $g^* dg = d \log g$.

$$\text{So } \mathfrak{f}(E)/_{G(E)^\circ} \cong Z^1(\Sigma)/B^1(\Sigma) = H^1(\Sigma).$$

The Betti moduli space is $\text{Hom}(\pi_1, \mathbb{C}^*) \cong \text{Hom}(\pi_1, S^1) \times \text{Hom}(\pi_1, \mathbb{R}^+)$

The de Rham moduli space:

$$\mathfrak{f}(E) = \mathfrak{f}_u(E) \times A^1(\Sigma, \mathbb{R})$$

$$D_0 + \eta \quad D_0 + i \text{Im} \eta \quad \text{Re} \eta$$

- $G(E) \cong \text{Map}(\Sigma, \mathbb{C}^*) = \text{Map}(\Sigma, S^1) \times \text{Map}(\Sigma, \mathbb{R}^+)$
 $\quad \quad \quad g \mapsto \begin{cases} \text{Map}(\Sigma, \mathbb{C}^*) \\ G_u(E) \end{cases} (g_u, g_r)$

$$\begin{aligned} g^*(D_0 + \eta) &= D_0 + \eta + g^{-1}dg \\ &= (D_0 + i\text{Im}\eta + (g_u^{-1}d\bar{g}_u)) + (\underbrace{(g_r^{-1}d\bar{g}_r + \text{Re}\eta)}_{d\log g_r}) \end{aligned}$$

- $\mathcal{F}(E)/G(E) \cong \mathcal{F}_u(E)/G_u(E) \times H^1(\Sigma, \mathbb{R})$.

$$\cong \left(\mathcal{F}_u(E)/\overset{\circ}{G_u(E)} \right) / \overline{\pi_0(G_u(E))} \times H^1(\Sigma, \mathbb{R})$$

(will see $\cong H^1(\Sigma, i\mathbb{R}) / \overline{H^1(\Sigma, \mathbb{Z})} \times H^1(\Sigma, \mathbb{R})$)

The equivalence between the moduli spaces is given by

$$\begin{aligned} h_{\text{op}} : \mathcal{F}(E) &\rightarrow \text{Hom}(\pi, \mathbb{C}^*) \\ D_0 + \eta &\mapsto (\gamma \mapsto \exp(\int_\gamma \eta)) \end{aligned}$$

Restrict to $h_{\text{op}} : \mathcal{F}_u(E) \rightarrow \text{Hom}(\pi, U(1))$

Claim: $\ker(h_{\text{op}}) = G(E)$.

Assuming the claim : h_{op} descends to a map

$$p : \left(\mathcal{F}_u(E)/G_u(E) \right) / \overline{\pi_0(G_u(E))} \rightarrow \begin{cases} \text{Hom}(\pi, U(1)) \\ \text{Hom}(\pi, i\mathbb{R}) \\ \text{Hom}(\pi, \mathbb{Z}) \end{cases}$$

$$\Rightarrow \pi_0(G_{\text{et}}(E)) \cong H^1(\Sigma, \mathbb{Z}).$$

$$\mathbb{Z}\langle w_1, \dots, w_{2g} \rangle$$

where w_1, \dots, w_{2g} has period $\in 2\pi i \mathbb{Z}$
and form a basis.

Pf of Claim:

" \Leftarrow " If $\eta \in H^1(\Sigma, i\mathbb{R})$ s.t. $\exp(\int_\gamma \eta) = 1 \quad \forall \gamma \in \Gamma$.

define $g(p) = \exp \int_{x_0}^p \eta$ well-defined.

as a fun $g: \Sigma \rightarrow \mathbb{C}^*$.

and $\eta = g^{-1}dg$.

" \Rightarrow " Given $g^{-1}dg$ for $g: \Sigma \rightarrow \mathbb{C}^*$,

one can lift g to $\tilde{g}: \overset{\circ}{\Sigma} \rightarrow \mathbb{C}^*$.

then $\tilde{g}^{-1}d\tilde{g}$ is exact on $\overset{\circ}{\Sigma}$,

$$d \log \tilde{g}.$$

Take a lift $\tilde{\gamma}$ of γ to $\overset{\circ}{\Sigma}$,

$$\begin{aligned} \int_\gamma g^{-1}dg &= \int_{\tilde{\gamma}(0)}^{\tilde{\gamma}(1)} \tilde{g}^{-1}d\tilde{g} = \int_{\tilde{\gamma}(0)}^{\tilde{\gamma}(1)} d \log \tilde{g} \\ &= \log \tilde{g}(\tilde{\gamma}(1)) - \log \tilde{g}(\tilde{\gamma}(0)) \\ &\in 2\pi i \mathbb{Z}. \end{aligned}$$

$$\Rightarrow \exp \int_\gamma g^{-1}dg = 1. \quad \square$$

Lecture 2

§5. The Dolbeault groupoid

Let X be a Riemann surface diffeo to Σ .

$$\mathcal{A}^1(X) = \frac{\mathcal{A}^{1,0}(X)}{dz} \oplus \frac{\mathcal{A}^{0,1}(X)}{d\bar{z}}$$

Hodge $*$ -operator on $\mathcal{A}^1(X)$

$$*dz = -\bar{z} dz$$

$$*d\bar{z} = \bar{z} d\bar{z}.$$

Defn. Given a complex vector bundle E over X ,

a holom str on E is a diff operator

$$\bar{\partial}_E : \mathcal{A}^{p,q}(X, E) \rightarrow \mathcal{A}^{p,q+1}(X, E)$$

which satisfy

$$\bar{\partial}_E(f \cdot s) = \bar{\partial}f \wedge s + f \cdot \bar{\partial}_E s.$$

$$\forall f \in \mathcal{A}^0(X), s \in \mathcal{A}^{p,q}(X, E).$$

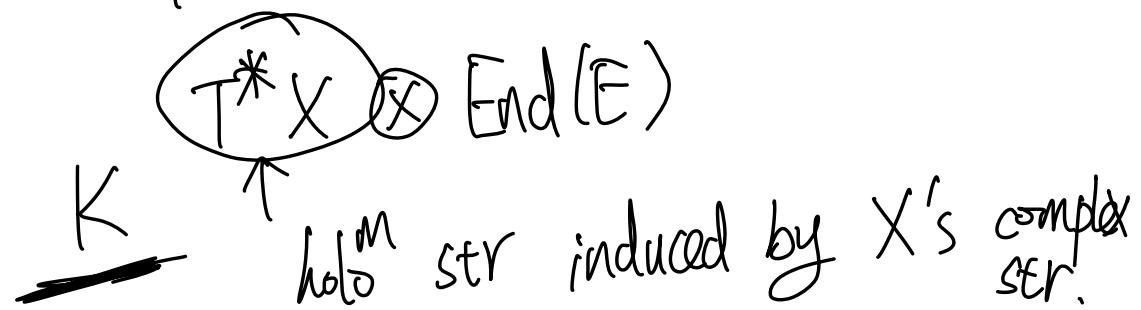
Rmk: If we are dealing w/ higher dim base mfld, we add the integrability

condition $\bar{\partial}_E^2 = 0$. of rk n

Defn. A Higgs bundle over X is
a pair (E, ϕ) where

- E is a hol^m v.b over X of rk n .
- ϕ is a hol^m 1-form on X taking values in $\underbrace{\text{End}(E)}$.
a hol^m v.b.

i.e. ϕ is a hol^m section of



$\underline{\phi}$ is called Higgs field.

Rmk: If the base mfld is of higher dim,
add the integrability condition $\Phi \wedge \underline{\Phi} = 0$.

- The $G(E)$ -action on the space of Higgs bundles is as follows:

$$(\bar{\partial}_E, \Phi) \xrightarrow{\xi} (\xi^* \bar{\partial}_E, \xi^* \phi)$$

\Downarrow " \Downarrow

$$\xi^* \bar{\partial}_E(\xi \cdot) \quad \xi^* \phi \xi.$$

Locally, w.r.t a frame e ,

$$\bar{\partial}_E e = e \Psi \quad (\bar{\partial}_E = \bar{\partial} + \Psi).$$

$$(\xi^* \bar{\partial}_E) e = e (g^{-1} \Psi g + g^{-1} \bar{\partial} g).$$

$g \leftrightarrow \xi.$

$$(\xi^* \phi) e = e \cdot g^{-1} \phi g.$$

Defn. The Dolbeault groupoid is

$$(Higgs(E), G(E))$$

↑
space of Higgs bundles over X .

§ 5.1 Understand the Dolbeault groupoid in $\text{rk } 1$, $\deg 0$ case.

E is trivial because $\text{rk } 1, \deg 0$ condition.

Start with a trivial complex line bundle E .

A Higgs field on the hol^M line bundle

$(E, \bar{\partial}_E)$ is just a hol^M 1-form on X .

since $\text{End}(E) = E \otimes E^* = \mathbb{C}$.

• Thus $\text{Higgs}(E) = \text{Hol}(E) \times \underline{\underline{H^{1,0}(X)}}$.

↑
space of hol^M str's on E

There is a standard base pt in $\text{Hol}(E)$.

$\bar{\partial}_0 = \bar{\partial}$ for $X \times \mathbb{C}$.

An arbitrary hol^M str on E is of the form $\bar{\partial}_0 + \Phi$,

where $\bar{\omega} \in \mathcal{A}^{0,1}(X)$.

- The gauge action $G(E)$ on $\text{Hol}(E)$
 $\bar{\omega} + \bar{\eta} \mapsto \bar{\omega} + \bar{\eta} + g^{-1} \bar{\partial} g.$

Again, $\frac{G(E)}{G(E)^0} = \pi_0(G(E))$

$$\frac{\text{Hol}(E)}{G(E)} = \left(\frac{\text{Hol}(E)}{G(E)^0} \right) / \pi_0(G(E))$$

$$G(E) = \text{Map}(X, \mathbb{C}^*)$$

$$G(E)^0 = \text{Map}(X, \mathbb{C}^*)^0 \quad (\text{containing constant maps})$$

$$g^E \quad g = \exp(f).$$

$$\text{So } \frac{\text{Hol}(E)}{G(E)^0} \cong \frac{\mathcal{A}^{0,1}(X)}{\text{exact}(0,1)\text{-form}} \cong \bar{\omega} \mathcal{A}^0(X).$$

By the Hodge decomposition

$$\text{Hol}(E) / G(E)^0 \cong H^{0,1}(X).$$

So the Dolbeault moduli space

$$\begin{aligned} \text{Higgs}(E) &\xrightarrow{\cong} \text{Hol}(E) \times H^{1,0}(X) \\ &\xrightarrow{\cong} H^{0,1}(X) \times H^{1,0}(X) \\ &\quad \underbrace{\qquad\qquad\qquad}_{T\mathcal{L}(G(E))} \end{aligned}$$

Claim: $T\mathcal{L}(G(E))$'s image form a lattice of rk 2g in $H^{0,1}(X)$.

(From last time, $T\mathcal{L}(G(E))$'s image in $H^1(X)$ is a lattice of rk 2g.)

$$\cong \text{Jac}(X) \times H^{1,0}(X).$$

π
a complex torus of dim g .

- Identify $\text{Higgs}(E)/_{G(E)}$ with $T^*\text{Jac}(X)$.

Consider the Hermitian form on $A(X)$

by $\langle \alpha, \beta \rangle := \int_X \alpha \wedge * \bar{\beta}$.

(pos. def on $A^{0,1}(X)$) $d\bar{z} \wedge -idz$.
 (neg. def on $A^{1,0}(X)$.)

Its restriction to $V = H^{0,1}(X)$

defines an isomorphism $\bar{V} \rightarrow V^*$.

of complex v.s.

$$\text{Higgs}(E)/_{G(E)} = \underbrace{H^{0,1}(X)}_{\text{Jac}(X)} \times \underbrace{H^{1,0}(X)}_{\bar{V}}$$

The tangent space of $\text{Jac}(X)$ at
any pt identifies with V .

Thus $V_L \times \bar{V} \cong V_L \times V^*$

$$\Rightarrow \text{Higgs}(E)/G(E) \cong T^*\text{Jac}(X).$$

§ 6. Equivalence between the
de Rham and Dolbeault groupoids
for $\text{rk } 1$, $\deg 0$ case.

§ 6.1. Introduce Hermitian metrics.

Defn. A Herm metric H on E is
a smooth family of pos. def Herm
forms $\langle , \rangle_H : E_x \times E_x \rightarrow \mathbb{C}$.

Denote by $\text{Her}(E)$ the space of Hermitian metrics on E .

In terms of a basis / frame e ,

$$H(\underline{e \cdot \beta}, \underline{e \cdot \eta}) = \xi^t h \eta \quad \text{Hermitian matrix.}$$

where $h = A(e, e)$

- The action of $G(E)$ on $\text{Her}(E)$,

locally, $g \cdot h = (g^{-1})^t h g^{-1}$.

- If E is a flat vector bundle on X with holonomy, then a Hermitian metric $\phi: \pi \rightarrow GL(r, \mathbb{C})$ corresponds to $H \in \text{Her}(E)$

a ϕ -equivariant map

$$h: X \longrightarrow \text{Her}(\mathbb{C}^r)$$

$$(\text{i.e. } h(\gamma \cdot x) = \phi(\gamma) h(x).)$$

Idea: Parallel transport H along paths based at x_0 to E_{x_0}

w.r.t this flat connection.

$$\text{i.e. } h([\gamma])^{(u,u)} = H(\underline{s(\gamma(i))}, s(\gamma(i)))$$

where $s(\gamma(t))$ is a parallel section along γ starting from u .

- Induced Hermitian pairing over $\mathcal{A}^*(X, E)$
 $\mathcal{A}^R(X, E) \times \mathcal{A}^L(X, E) \rightarrow \mathcal{A}^{RF}(X).$

Dfn. A connection D is unitary w.r.t H if

$$d\langle S_1, S_2 \rangle_H = \langle DS_1, S_2 \rangle_H + \langle S_1, DS_2 \rangle_H.$$

Prop. Given $(E, \bar{\partial}_E)$ with H ,

$\exists!$ a connection D s.t

$$(1) D^{0,1} = \bar{\partial}_E.$$

(2) D is unitary w.r.t H .

D is called Chern connection.

Prop. Given a connection D and H ,

$\exists!$ a decomposition

$$D = D_H + \tilde{\Psi}_H \quad \begin{matrix} \text{self-adjoint} \\ \text{w.r.t } H. \end{matrix}$$

s.t. unitary connection
w.r.t H .

$$\left(H(\tilde{\Psi}_H s, t) := \frac{1}{2} \{ H(Ds, t) + H(s, D(t)) \right. \\ \left. - d(H(s, t)) \} \right)$$

§ 6.2. Restrict to the case

E is a complex trivial line bundle.
with a frame τ .
trivialization.

Let H_0 be $H_0(\mathbb{C}, \mathbb{C}) = 1$.

$$G(E) = \text{Map}(X, \mathbb{C}^*) \ni g.$$

• $G(E)$ acts on $\text{Her}(E)$ as

$$h \mapsto |g|^2 h.$$

$$\left(\langle u, v \rangle_{g \cdot H} := \langle g^{-1}u, g^{-1}v \rangle_H \right)$$

$$h = H(\tau, \tau) : X \rightarrow \mathbb{R}^+$$

• $G(E)$ acts on $\text{Her}(E)$ transitively.

Want $g \cdot h_1 = h_2$,

need $g(z) = \sqrt{\frac{h_1(z)}{h_2(z)}}$.

- $D = D_0 + \eta$ is unitary w.r.t H
 iff

$$d(H(\tau, \tau)) = H(D\tau, \tau) + H(\tau, D\tau)$$

$$\Rightarrow dh = h\eta + h\bar{\eta}$$

$$\Rightarrow h^{-1}dh = \eta + \bar{\eta} = 2\operatorname{Re}(\eta).$$
- $\underline{\Psi}$ is self-adjoint w.r.t H
 iff $\underline{\Psi} = \bar{\underline{\Psi}}$ i.e. $\underline{\Psi}$ is real.

$$(H(\underline{\Psi}\tau, \tau) = H(\tau, \underline{\Psi}\tau))$$
- w.r.t H , $D = D_0 + \eta$ is uniquely
 decomposed $D = D_H + \underline{\Psi}_H$,

$$\begin{cases} D_H = D_0 + i\operatorname{Im}\eta + \frac{1}{2}h^{-1}dh. \\ \underline{\Psi}_H = \operatorname{Re}\eta - \frac{1}{2}h^{-1}dh. \end{cases}$$

§ 6.2(a). Start from a flat connection

Goal: To find a decomposition and further obtain a Higgs bundle.

Idea: Find the "best" H and use H to decompose.

Defn. A Hermitian metric h is harmonic w.r.t $D \in \mathcal{F}(E)$

if the associated equivariant map
 $h: X \rightarrow \text{Herm}(\mathbb{C}) = \mathbb{R}^+$.

(*) corresponding to h

is a multiplicatively harmonic fn.

(defined as, its logarithm is a harmonic fn.)

Prop. Condition (*) holds

iff $\bar{\omega}_H$ is a harmonic 1-form.
(iff $(\bar{\omega}_H)^{1,0}$ is hol^M.)

Pf: For a path $\gamma: [0,1] \rightarrow X$ with $\gamma(0) = x_0$.

Let s be the flat section from
 T along γ , then

$$(s(x_0) = T) \quad s(\gamma(1)) = \exp(-\int_{\gamma} \eta) \cdot T_0$$

$$\text{for } D = D_0 + \eta.$$

Use the defn of \hat{h} ,

$$h([x]) := H(s(\gamma(1)), s(\gamma(1)))$$

$$= H(\exp(-\int_{\gamma} \eta) \cdot T_0, \exp(\int_{\gamma} \eta) \cdot T_0)$$

$$\begin{aligned}
&= \exp(-\int_{\gamma}(\eta + \bar{\eta})) \cdot \underbrace{H(t_0, \bar{t}_0)}_{h} \\
&= \exp(-2\int_{\gamma} \operatorname{Re}(\eta) + h^T dh) \\
&= \exp(-2\int_{\gamma} (\operatorname{Re}(\eta) - \frac{1}{2}h^T dh)) \\
&= \exp(-2\int_{\gamma} \underline{\Psi_H}). \quad \square
\end{aligned}$$

Prop. For $D \in \mathcal{F}(E)$ $\exists!$ a harmonic metric H (up to constant scalar).

Pf: $D' = D_H + \underline{\Psi_H}$

$$\begin{aligned}
&= (D_0 + i \operatorname{Im} \eta + \frac{1}{2}h^T dh) \\
&\quad + \underline{(\operatorname{Re} \eta - \frac{1}{2}h^T dh)}
\end{aligned}$$

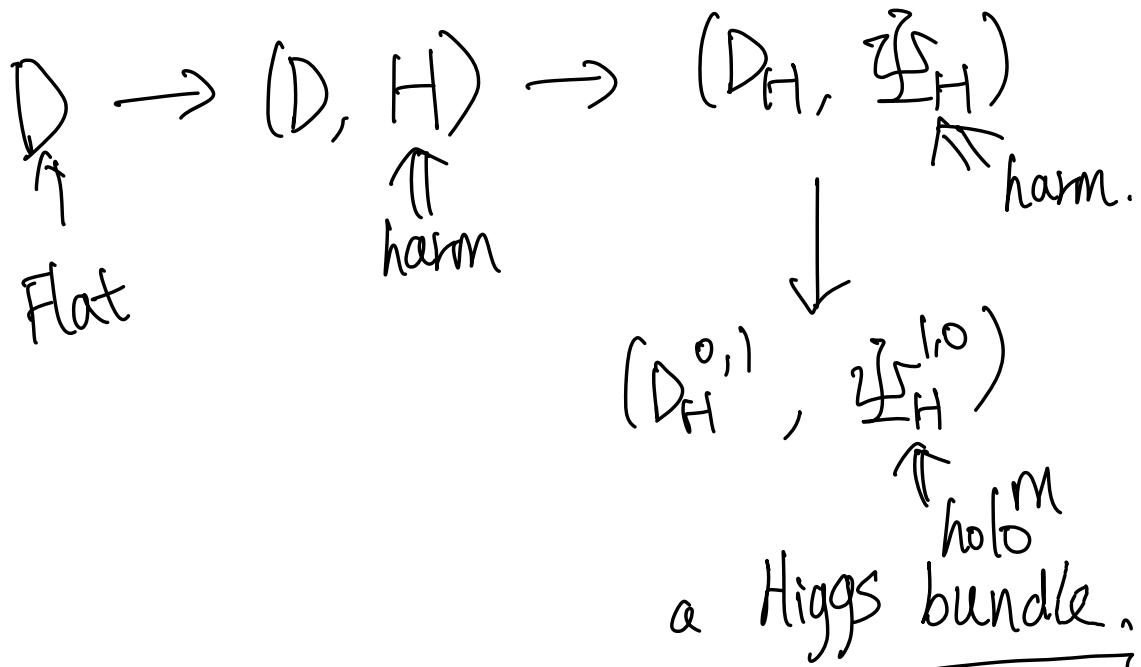
One wants to find h s.t

$\operatorname{Re} \eta - \frac{1}{2} h^{-1} dh$ is harmonic.

By Hodge decomposition,

$$\operatorname{Re} \eta = \text{harmonic 1-form} + \underline{dS}$$

Let $h = e^{2S}$.



§ 6.2(b). Start from a Higgs bundle.

One wants to find a way to sum up and obtain a flat connection.

Defn. A Hermitian metric is harmonic w.r.t $\bar{\partial}_E$ iff the Chern connection D^H is flat. iff $D^H + \phi + \bar{\phi}$ is flat.

Prop. For each $\bar{\partial}_E \in \text{Hol}(E)$,
 $\exists! h \in \text{Her}(E)$ up to constant s.t h is harmonic w.r.t $\bar{\partial}_E$.

Pf: Write $\bar{\partial}_E = \bar{\partial}_0 + \Psi$
 $= \bar{\partial}_0 + (\overset{C}{\Psi_0} + \bar{\Psi}_0)$
 $H^{0,1}(X)$

Note that h_0 is harmonic w.r.t $\bar{\partial}_0 + \overset{C}{\Psi_0}$,
 $(D^{h_0} = D_0 + \overset{C}{\Psi_0} - \bar{\Psi}_0.)$

Then $g \cdot h_0$ is harmonic to $\underline{g \cdot (\bar{\partial}_0 + \overset{C}{\Psi_0})}$.

Let $g = e^S$,

$$\bar{g}^!. (\bar{\partial}_0 + \bar{\partial}_S) = \bar{\partial}_0 + \bar{\partial}_S + \bar{\partial}S = \bar{\partial}E$$

So $\bar{g}^!. H_0$ is the desired metric.

□

$e^{2S} H_0$

§6.2(c)

Combine

Denote by $(\mathcal{F}(E) \times \text{Her}(E))_{\text{harm}}$ the subset

of (D, H) s.t. H is harm w.r.t D

Denote by $(\text{Higgs}(E) \times \text{Her}(E))_{\text{harm}}$ the subset

of $(\bar{\partial}_E, \phi, H)$ s.t. H is harm w.r.t $\bar{\partial}_E$.

We have a diagram:

$$\begin{array}{ccc}
 (\mathcal{F}(E), G(E)) & & (\text{Higgs}(E), G(E)) \\
 \downarrow & & \uparrow \\
 ((\mathcal{F}(E) \times \text{Herm}(E))_{\text{harm}}, G(E)) & \xrightarrow{(\mathcal{D}, H)} & ((\text{Higgs}(E) \times \text{Herm}(E))_{\text{harm}}, G(E)) \\
 & & \downarrow \\
 (\mathcal{D}^H + \phi + \bar{\phi}, H) & \longleftarrow & (\bar{\partial}_E, \phi, H)
 \end{array}$$

Thm. The induced functor
 $(\mathcal{F}(E), G(E)) \rightarrow (\text{Higgs}(E), G(E))$
is an equivalence of groupoids.

If: The rest need to check. 

S.T. Complex structures on moduli spaces. (again rk 1 cases.)

- Betti moduli space

$$M_{\text{Betti}} = \text{Hom}(\pi, \mathbb{C}^*) \cong (\mathbb{C}^*)^{2g}$$

$$T_p \text{Hom}(\pi, \mathbb{C}^*) \cong \text{Hom}(\pi, \mathbb{C}) = \mathbb{C}^{2g}.$$

$$\begin{aligned} J_1: T_p M_{\text{Betti}} &\rightarrow T_p M_{\text{Betti}} \\ x &\mapsto ix. \end{aligned}$$

- De Rham moduli space.

$$M_{\text{de Rham}} \cong \frac{H^1(X, i\mathbb{R})}{H^1(X, \mathbb{Z})} \times H^1(X, \mathbb{R})$$

$$\cong H^1(X) / H^1(X, \mathbb{Z})$$

$$T_x M_{\text{de Rham}} \cong H^1(X)$$

$$\begin{aligned} J_2: T_x M_{\text{de Rham}} &\rightarrow T_x M_{\text{de Rham}} \\ x &\mapsto ix. \end{aligned}$$

In fact, the equivalence between

$$M_{\text{de Rham}} \rightarrow M_{\text{Betti}}$$

$$D_0 + \eta \mapsto (\gamma \mapsto \exp \int_\gamma \eta)$$

The tangent map is
at η

$$X \mapsto (\gamma \mapsto \exp \int_\gamma \eta \cdot f_\gamma X)$$

is a biholo^m w.r.t J_1, J_2 .

So we can say $J_1 = J_2$, denoted by J .

- Dolbeault moduli space : $M_{\text{Dol}}, M_{\text{Higgs}}$

$$M_{\text{Dol}} = H^{0,1}(X) / \underbrace{\text{Jac}(X)}_{\sim} \times H^{1,0}(X)$$

$$T_\sigma M_{\text{Dol}} \cong H^{0,1}(X) \times H^{1,0}(X)$$

(±, ±)

I: $T_{\sigma}M_{Dol} \rightarrow \bar{T}_{\sigma}M_{Dol}$
 $(\underline{\psi}, \underline{\theta}) \mapsto (\bar{\psi}, \bar{\theta}).$

I is different from J.