Higgs bundles and related topics.

Qiongling Li

Chern Institute of Mathematics,
Nankai University

qiongling.li@nankai.edu.cn

Contents:

Lecture 1:

$\boxed{\text{Plan of the course}}$ Part I: Basics of Higgs bundles
geometry of moduli space 6
NAH
Higher Teichmüller theory
parabolic Higgs bundles

Part II: topics. 6.

Today: Explain Betti, de Rham, Dolbeault moduli spaces.
Take a close look at rk 1 case.

Reference:
W. Goldman and E.Z. Xia,
" Rank one Higgs bundles and representations of fundamental grps of R.S "

§0. Equivalence of deformation theories.

Defn. A deformation theory (or transformation groupoid) (S,G) consisting of a category $\mathcal{C}$ defined by a grp action as follows:

Let $\alpha: G\times S \to S$ left action.

(S,G) consists of the category $\mathcal{C}$ with $\mathrm{Obj}(\mathcal{C})=S$ with morphism $x \xrightarrow{g} y$ corresponding to the triple $(g, x, y) \in G \times S \times S$ s.t $a(g,x) = y$.

- $e \in G$ determines the identity morphism $x \xrightarrow{e} x$.
- $x \xrightarrow{g} y$ has an inverse $y \xrightarrow{g^{-1}} x$
- composition.

Defn. The moduli set corresponding to such a groupoid is the set $\mathrm{Iso}(\mathcal{L})$ of isomorphism classes of objects.

Defn. An equivalence of categories is a functor $F : \mathcal{A} \to \mathcal{B}$ s.t $\exists H : \mathcal{B} \to \mathcal{A}$ and
$$F \circ H \cong I_{\mathcal{B}}$$
$$H \circ F \cong I_{\mathcal{A}}.$$

$\rightsquigarrow$ a bijection : $\mathrm{Isom}(\mathcal{A}) \to \mathrm{Isom}(\mathcal{B})$.

Prop. (Criterion) A functor $F : \mathcal{A} \to \mathcal{B}$ is an equiv iff
(1) surjective on Isomorphism classes.
(2) Full : $F(x,y) : \mathrm{Mor}(x,y) \to \mathrm{Mor}(F(x), F(y))$ is surjective.
(3) Faithful : injective.

§1. $\boxed{\text{The Betti groupoid.}}$

Fix G a structure grp, e.g. $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $U(n)$;

Σ a compact smooth oriented surface with fundamental grp π.

- The objects are representations : $\pi \to G$
$$S = \mathrm{Hom}(\pi, G)$$
- The morphisms are from G by conjugation.
$$G \times \mathrm{Hom}(\pi, G) \longrightarrow \mathrm{Hom}(\pi, G)$$
$$g \cdot \rho \longmapsto g^{-1} \rho g$$

Defn. The Betti groupoid is $(\mathrm{Hom}(\pi, G), G)$.

- π admits a presentation

$$\langle A_1, B_1, \dots, A_g, B_g \mid [A_1, B_1]\cdots[A_g, B_g]=1]$$

The map

$$\begin{aligned} \mathrm{Hom}(\pi, G) &\hookrightarrow G^{2g} \\ \rho &\longmapsto (\rho(A_1), \rho(B_1), \dots, \rho(A_g), \rho(B_g)). \end{aligned}$$

embeds $\mathrm{Hom}(\pi, G)$ as a Zariski-closed subset of G^{2g} defined $[\alpha_1, \beta_1]\cdots[\alpha_g, \beta_g]=1$. $(*)$;

- If G is abelian, it acts trivially on $\mathrm{Hom}(\pi, G)$. The condition $(*)$ is automatically satisfied.

So

$$\begin{tikzcd} \mathrm{Hom}(\pi, G)/G \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\shortparallel"] & \mathrm{Hom}(\pi, G) \arrow[r, phantom, "\cong"] & G^{2g}. \\ \mathrm{Isom}((\mathrm{Hom}(\pi, G), G)). & & \end{tikzcd}$$

Will apply this to $G = \mathbb{C}^*, U(1), \mathbb{R}^+$.

§2. The de Rham groupoid

Let E be a smooth complex vector bundle over Σ.

$\mathcal{A}^k(\Sigma)$ denote the space of k-forms on Σ

$\mathcal{A}^k(\Sigma, E)$ . . . . E-valued k-forms.

Defn. A gauge transformation of E is a smooth bundle automorphism

$$\begin{tikzcd} E \arrow[r, "\xi"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d] \\ \Sigma \arrow[r, "\mathrm{id}"] & \Sigma \end{tikzcd}.$$

Denote by $\mathcal{G}(E)$ the group of gauge transformations of E.

__Defn__. (Connection)

A connection on E is an operator

$$D : \mathcal{A}^0(\Sigma; E) \longrightarrow \mathcal{A}^1(\Sigma, E)$$

s.t $\quad D(fs) = fD(s) + df \wedge Ds.$

Such a map extends to $D : \mathcal{A}^p(\Sigma; E) \to \mathcal{A}^{p+1}(\Sigma, E)$.

Denote by $u(E)$ the space of all connections on E.

Note that fix a connection D_0, an arbitrary connection

$$D = D_0 + \eta \quad \text{for} \quad \eta \in \mathcal{A}^1(\Sigma; \mathrm{End}(E)).$$

So $u(E)$ is an affine space modeled on $\mathcal{A}^1(\Sigma; \mathrm{End}(E))$.

__Defn__. (Curvature)

The curvature of a connection D is defined as $\quad F(D)s = D \circ D(s)$,

turns out to be an $\mathrm{End}(E)$-valued 2-form

$$F(D) \in \mathcal{A}^2(\Sigma; \mathrm{End}(E)).$$

Call D flat if $F(D) = 0$.

Denote by $\mathcal{F}(E)$ the space of flat connections on E.

(Note that for the existence of a flat connection, require $\deg(E) = 0$.)

- __The gauge action on connections__

$\xi^* D$ is defined as $\quad (\xi^* D)(s) = \xi^{-1} D(\xi \cdot s)$

for $\xi \in \mathcal{G}(E)$.

$$\xi \cdot D := (\xi^{-1})^* D.$$

Locally, w.r.t a frame e,

$$D = d + \eta \quad (\text{i.e } De = e\eta)$$

Then $\xi^* D = d + g^{-1}\eta g + g^{-1}dg$

(Here, g is the local expression of ξ w.r.t e.)

i.e. $\xi e = eg$.

$$(\xi^* D)(e) = D(eg) = e(\eta g + dg)$$
$$= eg(g^{-1}\eta g + g^{-1}dg).$$

- $F(\xi^* D) = \xi^*(F(D))$.

  Hence, $\mathcal{G}(E)$ preserves flatness.

Defn. The de Rham groupoid is $(\mathcal{F}(E), \mathcal{G}(E))$.

§3. Equivalence between Betti and de Rham groupoids

Start from a flat connection D on a vector bundle E, want to obtain a rep $\rho: \pi \to GL(n, \mathbb{C})$.

Locally, w.r.t a frame e, $De = e \cdot \eta$.

Over a smooth path $\gamma: [0,1] \to \Sigma$, parallel transport defines a linear map between the fibers $P_{\gamma(t)}: E_{\gamma(0)} \longrightarrow E_{\gamma(t)}$.

That is, $P_{\gamma(t)}(v)$ is parallel w.r.t D, for $v \in E_{\gamma(0)}$.

Suppose $v = (e\circ\gamma(0))\cdot f(0) \in E_{\gamma(0)}$.

Then $P_{\gamma(t)}(v) = (e\circ\gamma(t))\cdot \underline{g(t)}\cdot f(0)$ is parallel to D

$$\iff D_{\partial/\partial t}\big((e\circ\gamma(t))\cdot g(t)\cdot f(0)\big) = 0$$

$$\iff (e\circ\gamma(t))\,\underline{\underline{\big(\eta\circ\gamma(t)\cdot g(t) + dg(t)\big)}}^{(\partial/\partial t)}\cdot f(0) = 0$$

$$\iff g'(t) + (\eta\circ\gamma(t))\cdot g(t) = 0$$

$$\iff g(t) = \exp\left(-\int_0^t \gamma^*\eta\right)$$

<u>Fact</u>: Flatness of D implies the parallel transport only depends on homotopic class of γ relative to its endpts.

Now we obtain a homomorphism: fix a pt $p \in E_{x_0}$.

$$\mathrm{hol}_p(D) : \pi_1(X, x_0) \longrightarrow GL(n,\mathbb{C})$$
$$\gamma \longmapsto \Big(P_\gamma : E_{\underset{\gamma(0)}{\overset{\shortparallel}{x_0}}} \to E_{\underset{\gamma(1)}{\overset{\shortparallel}{x_0}}}\Big)^{-1}$$

w.r.t a fixed frame e at E_{x_0}.

<u>Thm</u>: The holonomy functor
$$\mathrm{hol}_p : (\mathcal{F}(E), \mathcal{G}(E)) \longrightarrow (\mathrm{Hom}(\pi, GL(n,\mathbb{C})), GL(n,\mathbb{C}))$$
is an equivalence of groupoids.

Pf: • surjective on isomorphism classes.

Given a rep $\rho \in \mathrm{Hom}(\pi, GL(n,\mathbb{C}))$, we construct a flat vector bundle $\mathbb{C}_\rho \to \Sigma$ as follows:

the grp π acts on the total space $\check{\Sigma}\times\mathbb{C}^n$ by

$$\gamma\cdot(\check{s}, x) := (\underset{\substack{\uparrow\\ \text{deck transformation}}}{\overset{``\check{s}\cdot\gamma^{-1}}{\gamma\cdot\check{s}}}, \underline{\rho(\gamma)}x) \quad \forall \gamma\in\pi.$$

The quotient $(\check{\Sigma}\times\mathbb{C}^n)/\pi$ is the total space of a smooth vector bundle $\mathbb{C}_\rho \xrightarrow{p} \Sigma$, which carries a natural flat connection D as the descending of $D_0 = d$ on $\check{\Sigma}\times\mathbb{C}^n$.

$[(\check{\Sigma}, \underset{\substack{\uparrow\\ \text{constant}}}{v})]$ is parallel to D.

So this D gives holonomy ρ up to conjugation. ▨

– Full and faithful (need to check.)

§4. Rank 1 case for equivalence between Betti and de Rham moduli spaces.

Let E be a trivial complex line bundle over Σ.

A trivialization τ is a global frame of E.

- The gauge transformation $\xi\in\mathcal{G}(E)$ is determined by a smooth map $g:\Sigma\to\mathbb{C}^*$ via

$$\xi(\tau) = g\cdot\tau.$$

$$\mathcal{G}(E)\cong \mathrm{Map}(\Sigma,\mathbb{C}^*).$$

The subgrp $\mathcal{G}_u(E)\cong \mathrm{Map}(\Sigma, U(1))$

Let $\mathrm{Map}(\Sigma,\mathbb{C}^*)^\circ$ denote the component containing the constant map.

$\mathcal{G}(E)^\circ =$

$$G(E)/G(E)^\circ = \pi_0(G(E))$$

Note that $\mathrm{Map}(\Sigma, \mathbb{C}^*)^\circ \cong \mathcal{A}^0(\Sigma)$

$$g \longmapsto \log g .$$

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ \Sigma \arrow[ur, dashed, "\log g"] \arrow[r, "g"'] & \mathbb{C}^* \end{tikzcd}$$

iff $g_*: \pi_1 \Sigma \to \pi_1(\mathbb{C}^*)$ is trivial

So $$\mathcal{F}(E)/G(E) = \underline{\underline{\left(\mathcal{F}(E)/G(E)^\circ\right)}} \Big/ \underline{\pi_0(G(E))} .$$

- On E, there is a unique connection D_0 s.t $D_0 \tau = 0$.

  Any connection D is of the form
  $$D = D_0 + \eta, \quad \eta \in \mathcal{A}^1(\Sigma).$$
  D is flat $\Leftrightarrow$ $d\eta = 0$.

  $\xi^*(D_0 + \eta) = D_0 + \eta + g^{-1}dg$. $(\xi \leftrightarrow g \in \mathrm{Map}(\Sigma, \mathbb{C}^*$

  If $g \in \mathrm{Map}(\Sigma, \mathbb{C}^*)^\circ$, $g^{-1}dg = d \log g$.

  So $$\mathcal{F}(E)/G(E)^\circ \cong Z^1(\Sigma)/B^1(\Sigma) = H^1(\Sigma).$$

The Betti moduli space is $\mathrm{Hom}(\pi, \mathbb{C}^*) \cong \mathrm{Hom}(\pi, S^1) \times \mathrm{Hom}(\pi, \mathbb{R}^+)$

The de Rham moduli space:

- $$\underset{D_0 + \eta}{\mathcal{F}(E)} = \underset{D_0 + i\,\mathrm{Im}\,\eta}{\mathcal{F}_u(E)} \times \underset{\mathrm{Re}\,\eta}{\mathcal{A}^1(\Sigma, \mathbb{R})}$$

- $$\mathcal{G}(E) \cong \mathrm{Map}(\Sigma, \mathbb{C}^*) = \underset{\mathcal{G}_u(E)}{\underset{\shortparallel}{\mathrm{Map}(\Sigma, S^1)}} \times \mathrm{Map}(\Sigma, \mathbb{R}^+)$$
$$g \mapsto (g_u, g_r)$$

$$g^*(D_0+\eta) = D_0 + \eta + g^{-1}dg$$
$$= (D_0 + i\mathrm{Im}\eta + \underline{(g_u^{-1}\overline{dg}_u)}) + (\underbrace{g_r^{-1}d\bar{g}_r + \mathrm{Re}\eta}_{\underset{d\log g_r}{\shortparallel}})$$

- $$\mathcal{F}(E)/\mathcal{G}(E) \cong \mathcal{F}_u(E)/\mathcal{G}_u(E) \times H^1(\Sigma, \mathbb{R}).$$
$$\cong \underset{H^1(\Sigma, i\mathbb{R})}{\underline{\underline{\left(\mathcal{F}_u(E)/\mathcal{G}_u^0(E)\right)}}} \Big/ \pi_0(\mathcal{G}_u(E)) \times H^1(\Sigma, \mathbb{R})$$

(will see $\cong H^1(\Sigma, i\mathbb{R}) \big/ H^1(\Sigma, \mathbb{Z}) \times H^1(\Sigma, \mathbb{R})$.)

The equivalence between the moduli spaces is given by

$$hol_p : \mathcal{F}(E) \longrightarrow \mathrm{Hom}(\pi, \mathbb{C}^*)$$
$$D_0 + \eta \longmapsto (\gamma \mapsto \exp(\textstyle\int_\gamma \eta))$$

Restrict to $hol_p : \mathcal{F}_u(E) \longrightarrow \mathrm{Hom}(\pi, U(1))$

<u>Claim:</u> $\ker(hol_p) = \mathcal{G}(E)$.

Assuming the claim: hol_p desends to a map

$$p : \underbrace{\left(\mathcal{F}_u(E)/\mathcal{G}_u(E)\right)}_{\underset{H^1(X, i\mathbb{R})}{\shortparallel}} \Big/ \underline{\pi_0(\mathcal{G}_u(E))} \longrightarrow \underset{\mathrm{Hom}(\pi, i\mathbb{R}) \big/ \underline{\mathrm{Hom}(\pi, 2\pi i\mathbb{Z})}}{\underset{\shortparallel}{\mathrm{Hom}(\pi, U(1))}}$$

$$\Rightarrow \pi_0(G_u(E)) \cong H^1(\Sigma, \mathbb{Z}).$$

$$\overset{\shortparallel}{\mathbb{Z}\langle \omega_1, \cdots, \omega_{2g}\rangle}$$

where $\omega_1, \cdots, \omega_{2g}$ has period $\in 2\pi i \mathbb{Z}$ and form a basis.

<u>Pf of Claim:</u>

"$\subset$" If $\eta \in H^1(\Sigma, i\mathbb{R})$ s.t $\exp(\int_\gamma \eta) = 1 \quad \forall \gamma \in \pi$.

define $g(p) = \exp \int_{x_0}^{p} \eta$ well-defined.

as a fcn $g : \Sigma \to \mathbb{C}^*$.

and $\eta = g^{-1} dg$.

"$\supset$" Given $g^{-1}dg$ for $g : \Sigma \to \mathbb{C}^*$,

one can lift g to $\tilde{g} : \tilde{\Sigma} \to \mathbb{C}^*$.

then $\underset{\overset{\shortparallel}{d \log \tilde{g}}}{\tilde{g}^{-1} d\tilde{g}}$ is exact on $\tilde{\Sigma}$.

Take a lift $\tilde{\gamma}$ of γ to $\tilde{\Sigma}$,

then
$$\int_\gamma g^{-1}dg = \int_{\tilde{\gamma}(0)}^{\tilde{\gamma}(1)} \tilde{g}^{-1} d\tilde{g} = \int_{\tilde{\gamma}(0)}^{\tilde{\gamma}(1)} d \log \tilde{g}$$
$$= \log \tilde{g}(\tilde{\gamma}(1)) - \log \tilde{g}(\tilde{\gamma}(0))$$
$$\in 2\pi i \mathbb{Z}.$$

$$\Rightarrow \exp \int_\gamma g^{-1} dg = 1.$$ ▨

$\boxed{\text{Lecture 2}}$

§5. The Dolbeault groupoid

Let X be a Riemann surface diffeo to Σ.

$$\mathcal{A}^1(X) = \underset{dz}{\mathcal{A}^{1,0}(X)} \oplus \underset{d\bar{z}}{\mathcal{A}^{0,1}(X)}$$

Hodge $*$-operator on $\mathcal{A}^1(X)$

$$*dz = -i dz$$
$$*d\bar{z} = i d\bar{z}.$$

Defn. Given a complex vector bundle E over X, a holo^m str on E is a diff operator

$$\bar{\partial}_E : \mathcal{A}^{p,q}(X,E) \to \mathcal{A}^{p,q+1}(X,E)$$

which satisfy

$$\bar{\partial}_E(f \cdot s) = \bar{\partial} f \wedge s + f \cdot \bar{\partial}_E s.$$
$$\forall f \in \mathcal{A}^0(X), \quad s \in \mathcal{A}^{p,q}(X,E).$$

Rmk: If we are dealing w/ higher dim base mfld, we add the integrability condition $\bar{\partial}_E^2 = 0$.

<u>Defn</u>. A Higgs bundle over X $\overset{\text{of rk } n}{\vee}$ is a pair (E, Φ) where

- E is a holo^m v.b over X of rk n.
- Φ is a holo^m 1-form on X taking values in $\underbrace{\mathrm{End}(E)}_{\text{a holo}^m \text{ v.b.}}$.

i.e. Φ is a holo^m section of

$$\widehat{T^*X} \otimes \mathrm{End}(E)$$

K $\uparrow$ holo^m str induced by X's complex str.

Φ is called Higgs field.

<u>Rmk</u>: If the base mfld is of higher dim, add the integrability condition $\Phi \wedge \Phi = 0$.

- The $\mathcal{G}(E)$-action on the space of Higgs bundles is as follows:

$$(\bar\partial_E, \Phi) \xrightarrow{\xi} \Big(\underset{\xi^{-1}\bar\partial_E(\xi\,\bullet)}{\underset{\|}{\xi^*\bar\partial_E}},\ \underset{\xi^{-1}\phi\,\xi.}{\underset{\|}{\xi^*\phi}}\Big)$$

Locally, w.r.t a frame e,

$$\bar\partial_E e = e\Phi \qquad \underline{(\bar\partial_E = \bar\partial + \Phi).}$$

$$(\xi^*\bar\partial_E)e = e(g^{-1}\Phi g + g^{-1}\bar\partial g).$$

$$g \leftrightarrow \xi.$$

$$(\xi^*\phi)e = e\cdot g^{-1}\phi g.$$

<u>Defn</u>. The Dolbeault groupoid is

$$\big(\ \underset{\substack{\uparrow \\ \text{space of Higgs bundles over } X.}}{\mathrm{Higgs}(E)},\ \mathcal{G}(E)\big)$$

§5.1 Understand the Dolbeault groupoid in rk1, deg 0 case.

E is trivial because rk 1, deg 0 condition.

Start with a trivial complex line bundle E.

A Higgs field on the holo^{m} line bundle $(E, \bar\partial_E)$ is just a holo^{m} 1-form on X.

since $\mathrm{End}(E) = E \otimes E^{*} = \mathcal{O}$.

- Thus $\mathrm{Higgs}(E) = \underset{\Uparrow}{\mathrm{Hol}(E)} \times \underline{\underline{H^{1,0}(X).}}$

space of holo^{m} str's on E

There is a standard base pt in Hol(E).

$\bar\partial_0 = \bar\partial$ for $X \times \mathbb{C}$.

An arbitrary holo^{m} str on E is of the form $\bar\partial_0 + \Phi$, where $\Phi \in \mathcal{A}^{0,1}(X)$.

- The gauge action $\mathcal{G}(E)$ on $\mathcal{H}ol(E)$

$$\bar{\partial}_0 + \Phi \longmapsto \bar{\partial}_0 + \Phi + \underbrace{g^{-1}\bar{\partial} g}.$$

Again, $\mathcal{G}(E) / \mathcal{G}(E)^0 = \pi_0(\mathcal{G}(E))$

$$\mathcal{H}ol(E) / \mathcal{G}(E) = \left(\mathcal{H}ol(E) / \mathcal{G}(E)^0 \right) \Big/ \pi_0(\mathcal{G}(E))$$

$\mathcal{G}(E) = \mathrm{Map}(X, \mathbb{C}^*)$

$\mathcal{G}(E)^0 = \mathrm{Map}(X, \mathbb{C}^*)^0$ (containing constant maps)

$g \in$ $\quad g = \exp(f)$.

So $\mathcal{H}ol(E) / \mathcal{G}(E)^0 \cong \mathcal{A}^{0,1}(X) \Big/$ exact (0,1)-form $\bar{\partial}\mathcal{A}^0(X)$.

By the Hodge decomposition

$$\mathrm{Hol}(E)/\mathcal{G}(E)^0 \cong H^{0,1}(X).$$

So the Dolbeault moduli space

$$\mathrm{Higgs}(E)/\mathcal{G}(E) \cong \mathrm{Hol}(E)/\mathcal{G}(E) \times H^{1,0}(X)$$

$$\cong H^{0,1}(X)/\underbrace{\pi_0(\mathcal{G}(E))} \times H^{1,0}(X).$$

Claim: $\pi_0(\mathcal{G}(E))$'s image form a lattice of rk $2g$ in $H^{0,1}(X)$.

(From last time, $\pi_0(\mathcal{G}(E))$'s image in $H^1(X)$ is a lattice of rk $2g$.)

$$\simeq \underset{\substack{\Uparrow \\ \text{a complex torus of dim } g.}}{\mathrm{Jac}(X)} \times H^{1,0}(X).$$

- Identify $\mathrm{Higgs}(E)/G(E)$ with $T^*\mathrm{Jac}(X)$.

Consider the Hermitian form on $\mathcal{A}^1(X)$

by $\langle \alpha, \beta \rangle := \int_X \alpha \wedge * \bar{\beta}$.

$$\begin{pmatrix} \text{pos.def on } \mathcal{A}^{0,1}(X) \\ \text{neg.def on } \mathcal{A}^{1,0}(X). \end{pmatrix} \quad d\bar{z} \wedge -i\,dz.$$

Its restriction to $V = H^{0,1}(X)$
defines an isomorphism $\bar{V} \longrightarrow V^*$.
of complex v.s.

$$\mathrm{Higgs}(E)/G(E) = \underset{\substack{\wr\| \\ \mathrm{Jac}(X)}}{\boxed{H^{0,1}(X)}\big/\overset{\cup}{L}} \times \underbrace{H^{1,0}(X)}_{\substack{\circ \\ \bar{V}}}$$

The tangent space of $\mathrm{Jac}(X)$ at any pt identifies with V.

Thus $V/L \times \bar{V} \cong V/L \times V^*$

$\Rightarrow \mathrm{Higgs}(E)/\mathcal{G}(E) \cong T^*\mathrm{Jac}(X)$.

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§6. Equivalence between the de Rham and Dolbeault groupoids for rk 1, deg 0 case.

§6.1. Introduce Hermitian metrics.

Defn. A Herm metric H on E is a smooth family of pos. def Herm forms $\langle , \rangle_H : E_x \times E_x \to \mathbb{C}$.

Denote by Her(E) the space of Herm metrics on E.

In terms of a basis / frame e,

$$H(\underline{e \cdot \xi}, \underline{e \cdot \eta}) = \xi^t h \eta$$ Hermitian

where $h = H(e, e)$ matrix.

- The action of $G(E)$ on Her(E), locally, $g \cdot h = (g^{-1})^t h g^{-1}$.

- If E is a flat vector bundle on X with holonomy $\phi : \pi \to GL(r, \mathbb{C})$, then a Hermitian metric $H \in$ Her (E) corresponds to

a ϕ - equivariant map

$$h : \hat{X} \longrightarrow Her(\mathbb{C}^r)$$

(i.e. $h(\gamma\cdot x) = \phi(\gamma)\, h(x)$.)

Idea: Parallel transport H along paths based at x_0 to E_{x_0} w.r.t this flat connection.

i.e. $h([\gamma])^{(u,u)} = H(s(\underline{\gamma(1)}), s(\gamma(1)))$

where $s(\gamma(t))$ is a parallel section along γ starting from u.

- Induced Hermition pairing over $\mathcal{A}^*(X,E)$

$$\mathcal{A}^k(X,E) \times \mathcal{A}^l(X,E) \to \mathcal{A}^{k+l}(X).$$

<u>Defn</u>. A connection D is unitary w.r.t H if

$$d\langle s_1, s_2\rangle_H = \langle Ds_1, s_2\rangle_H + \langle s_1, Ds_2\rangle_H.$$

Prop. Given $(E, \bar{\partial}_E)$ with H,
$\exists!$ a connection D s.t
(1) $D^{0,1} = \bar{\partial}_E$.
(2) D is unitary w.r.t H.

D is called Chern connection.

Prop. Given a connection D and H,
$\exists!$ a decomposition

$$D = \underset{\substack{\uparrow \\ \text{s.t unitary connection} \\ \text{w.r.t } H.}}{D_H} + \Phi_H \nwarrow \text{ self-adjoint w.r.t } H.$$

$$\left(H(\Phi_H s, t) := \frac{1}{2}\{H(Ds, t) + H(s, D(t)) - d(H(s,t))\}. \right)$$

§6.2. Restrict to the case

E is a complex trivial line bundle

with a frame τ.

trivialization .

Let H_0 be $H_0(\tau, \tau) = 1$.

$$G(E) = \mathrm{Map}(X, \mathbb{C}^*) \ni g.$$

- $G(E)$ acts on $\mathrm{Her}(E)$ as

$$h \longmapsto |g|^{-2} h.$$

$$\left(\langle u, v \rangle_{g \cdot H} := \langle g^{-1} u, g^{-1} v \rangle_H, \quad h = H(\tau, \tau) : X \to \mathbb{R}^+. \right)$$

- $G(E)$ acts on $\mathrm{Her}(E)$ transitively.

Want $g \cdot h_1 = h_2$,

need $g(z) = \sqrt{h_1(z)/h_2(z)}$.

- $D = D_0 + \eta$ is unitary w.r.t H

  iff

  $$d(H(\tau,\tau)) = H(D\tau,\tau) + H(\tau,D\tau)$$

  $$\Rightarrow \quad dh = h\eta + h\bar{\eta}$$

  $$\Rightarrow \quad h^{-1}dh = \eta + \bar{\eta} = 2\mathrm{Re}(\eta).$$

- Φ is self-adjoint w.r.t H

  iff $\Phi = \bar{\Phi}$ i.e. Φ is real.

  $$\left(H(\Phi\tau,\tau) = H(\tau,\Phi\tau) \right)$$

- w.r.t H, $D = D_0 + \eta$ is uniquely decomposed $D = D_H + \Phi_H$,

  $$\begin{cases} D_H = D_0 + i\,\mathrm{Im}\eta + \frac{1}{2}h^{-1}dh. \\ \Phi_H = \mathrm{Re}\eta - \frac{1}{2}h^{-1}dh. \end{cases}$$

§6.2(a). $\boxed{\text{Start from a flat connection}}$

Goal: To find a decomposition and further obtain a Higgs bundle.

Idea: Find the "best" H and use H to decompose.

<u>Defn</u>. A Hermitian metric h is <u>harmonic</u> w.r.t $D \in \mathcal{F}(E)$ if the associated equivariant map

$$\check{h} : \check{X} \to \mathrm{Herm}(\mathbb{C}) = \mathbb{R}^+.$$

(*) corresponding to h is a <u>multiplicatively harmonic fcn</u>.

(~~defined~~ as, its logarithm is a harmonic fcn.)

Prop. Condition (*) holds

iff $\partial\Xi_H$ is a harmonic 1-form.

(iff $(\partial\Xi_H)^{1,0}$ is holo^{m}.)

Pf: For a path $\gamma:[0,1]\to X$ with $\gamma(0)=x_0$.

Let s be the flat section from τ along γ, then

$(s(x_0)=\tau)$

$$s(\gamma(1)) = \exp\left(-\int_\gamma \eta\right)\cdot \tau_0$$

for $D = D_0 + \eta$.

Use the defn of $\breve{h}$,

$$\breve{h}(\underset{\underset{\breve{X}}{\uparrow}}{[\gamma]}) := H(s(\gamma(1)), s(\gamma(1)))$$

$$= H\left(\exp\left(-\int_\gamma \eta\right)\cdot \tau_0, \exp\left(-\int_\gamma \eta\right)\cdot \tau_0\right)$$

$$= \exp\left(-\int_\gamma (\eta + \bar{\eta})\right) \cdot \underbrace{H(\tau_0, \tau_0)}_{h}$$

$$= \exp\left(-2\int_\gamma \mathrm{Re}(\eta) + h^{-1}dh\right)$$

$$= \exp\left(-2\int_\gamma \left(\mathrm{Re}(\eta) - \tfrac{1}{2}h^{-1}dh\right)\right)$$

$$= \underline{\underline{\exp\left(-2\int_\gamma \Phi_H\right)}}. \quad ▨$$

<u>Prop</u>. For $D \in \mathcal{F}(E)$, $\exists!$ a harmonic metric H (up to constant scalar).

Pf: $D \overset{\text{"}D_0+\eta}{=} D_H + \Phi_H$

$$= \left(D_0 + i\,\mathrm{Im}\,\eta + \tfrac{1}{2}h^{-1}dh\right)$$
$$+ \underline{\underline{\left(\mathrm{Re}\,\eta - \tfrac{1}{2}h^{-1}dh\right)}}$$

One wants to find h s.t

$\operatorname{Re}\eta - \frac{1}{2}h^{-1}dh$ is harmonic.

By Hodge decomposition,

$$\operatorname{Re}\eta = \text{harmonic 1-form} + \underline{\underline{ds}}$$

Let $h = e^{2s}$.

▨

$$\begin{tikzcd}
D \arrow[r] & (D, H) \arrow[r] & (D_H, 2\Phi_H) \arrow[d] & \\
\text{Flat} \arrow[u] & \text{harm} \arrow[u, Rightarrow] & (D_H^{0,1}, 2\Phi_H^{1,0}) & \text{harm.} \arrow[ul, Rightarrow] \\
 & & \text{holom} \arrow[u, Rightarrow] &
\end{tikzcd}$$

a Higgs bundle.

§6.2(b). $\boxed{\text{Start from a Higgs bundle}}$

One wants to find a way to sum up and obtain a flat connection.

Defn. A Hermition metric is harmonic w.r.t $\bar{\partial}_E$ iff the Chern connection D^H is flat.

iff $D^H + \phi + \bar{\phi}$ is flat.

Prop. For each $\bar{\partial}_E \in Hol(E)$, $\exists!$ $h \in Her(E)$ up to constant s.t h is harmonic w.r.t $\bar{\partial}_E$.

Pf: Write

$$\bar{\partial}_E = \bar{\partial}_0 + \Psi = \bar{\partial}_0 + (\underset{\mathcal{H}^{0,1}(X)}{\underset{\uparrow}{\Psi_0}} + \partial S)$$

Note that H_0 is harmonic w.r.t $\bar{\partial}_0 + \Psi_0$,

$(D^{H_0} = D_0 + \Psi_0 - \bar{\Psi}_0.)$

Then $g \cdot H_0$ is harmonic to $\underline{\underline{g \cdot (\bar{\partial}_0 + \Psi_0)}}$.

Let $g = e^{s}$,

$$g^{-1}\cdot(\bar\partial_0 + \Phi_0) = \bar\partial_0 + \Phi_0 + \bar\partial s = \bar\partial_E$$

So $g^{-1}\cdot H_0$ is the desired metric.

$\boxtimes$

$\overset{\shortparallel}{e^{2s} H_0}$

§6.2(c) $\boxed{\text{Combine}}$

Denote by $(\mathcal{F}(E) \times \mathrm{Her}(E))_{\mathrm{harm}}$ the subset of (D, H) s.t H is harm w.r.t D

Denote by $(\mathrm{Higgs}(E) \times \mathrm{Her}(E))_{\mathrm{harm}}$ the subset of $(\bar\partial_E, \phi, H)$ s.t H is harm w.r.t $\bar\partial_E$.

We have a diagram:

$$\begin{tikzcd}
(\mathcal{F}(E), \mathcal{G}(E)) \arrow[d, leftrightarrow] & & (\mathrm{Higgs}(E), \mathcal{G}(E)) \arrow[d, leftrightarrow] \\
((\mathcal{F}(E) \times \mathrm{Herm}(E))_{\mathrm{harm}}, \mathcal{G}(E)) & & ((\mathrm{Higgs}(E) \times \mathrm{Herm}(E))_{\mathrm{harm}}, \mathcal{G}(E))
\end{tikzcd}$$

$$(D, H) \longmapsto (D_H^{0,1}, \Phi_H^{1,0}, H)$$

$$(D^H + \phi + \bar{\phi}, H) \longleftarrow\!\shortmid (\bar{\partial}_E, \phi, H)$$

<u>Thm</u>. The induced functor
$$(\mathcal{F}(E), \mathcal{G}(E)) \longrightarrow (\mathrm{Higgs}(E), \mathcal{G}(E))$$
is an equivalence of groupoids.

Pf: The rest need to check. ▨

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§7. Complex structures on moduli spaces. (again rk 1 cases.)

- Betti moduli space

$$\mathcal{M}_{\text{Betti}} = \mathrm{Hom}(\pi, \mathbb{C}^*) \simeq (\mathbb{C}^*)^{2g}$$

$$T_\rho \mathrm{Hom}(\pi, \mathbb{C}^*) \simeq \mathrm{Hom}(\pi, \mathbb{C}) = \mathbb{C}^{2g}.$$

$$J_1 : T_\rho \mathcal{M}_{\text{Betti}} \longrightarrow T_\rho \mathcal{M}_{\text{Betti}}$$

$$X \longmapsto iX.$$

- De Rham moduli space.

$$\mathcal{M}_{\text{de Rham}} \simeq H^1(X, i\mathbb{R}) \Big/ H^1(X, \mathbb{Z}) \times H^1(X, \mathbb{R})$$

$$\simeq H^1(X) \big/ H^1(X, \mathbb{Z})$$

$$T_\eta \mathcal{M}_{\text{de Rham}} \simeq H^1(X)$$

$$J_2 : T_\eta \mathcal{M}_{\text{de Rham}} \longrightarrow T_\eta \mathcal{M}_{\text{de Rham}}$$

$$X \longmapsto iX.$$

In fact, the equivalence between

$$\mathcal{M}_{\text{de Rham}} \longrightarrow \mathcal{M}_{\text{Betti}}$$

$$D_0 + \eta \longmapsto \left(\gamma \mapsto \exp \int_\gamma \eta\right)$$

The tangent map at η is $X \longmapsto \left(\gamma \mapsto \exp\int_\gamma \eta \cdot \int_\gamma X\right)$

is a biholo^m w.r.t J_1, J_2.

So we can say $J_1 = J_2$, denoted by J.

- <u>Dolbeault moduli space</u> : $\mathcal{M}_{\text{Dol}}$, $\mathcal{M}_{\text{Higgs}}$

$$\mathcal{M}_{\text{Dol}} = \underbrace{H^{0,1}(X)/L}_{\text{Jac}(X)} \times H^{1,0}(X)$$

$$T_\sigma \mathcal{M}_{\text{Dol}} \cong \underset{(\Phi,\ \Phi)}{H^{0,1}(X) \times H^{1,0}(X)}$$

$$I : T_\sigma M_{Dol} \longrightarrow T_\sigma M_{Dol}$$

$$(\dot{A}, \dot{\Phi}) \longmapsto (i\dot{A}, i\dot{\Phi}).$$

I is different from J.

Lecture 3

X — a compact R.S of $g \geq 2$ if not specified.

$\mathcal{E}$ — $(E, \bar{\partial}_E)$ holo^m v.b.

Goal today:

- moduli space of polystable vector bundles / Higgs bundles
- relate the stability with soln to Hitchin eqn.

§1. Preparation: extensions of holo^m vector bundles.

If $\mathcal{F} \subset \mathcal{E}$ is a holo^m subbundle, then $Q = \mathcal{E}/\mathcal{F}$ has an induced holo^m str, $\bar{\partial}_Q$.

Moreover, we can choose a complement subbundle of $\mathcal{F}$ insider $\mathcal{E}$ to represent Q.
(e.g. choose $Q = \mathcal{F}^{\perp_H}$)

Write $\mathcal{E} = \mathcal{F} \oplus Q$ C^∞ direct sum

Then
$$\bar{\partial}_{\mathcal{E}} = \begin{pmatrix} \bar{\partial}_{\mathcal{F}} & \beta \\ 0 & \bar{\partial}_Q \end{pmatrix},$$

where $\beta \in A^{0,1}(X, \mathrm{Hom}(Q, \mathcal{F}))$;

called the secondamental form.

- If β is in $\bar{\partial}(\underline{A^0}(\underline{X, \mathrm{Hom}}(\underline{Q}, \mathcal{F})))$, then by a gauge transformation,

  $\mathcal{E} = \mathcal{F} \oplus Q'$ holo^m splitting.

- Isomorphism classes of $\mathcal{E}$ of the form

$$\begin{tikzcd} 0 \arrow[r, hook] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & Q \arrow[r] \arrow[l, dashed, hook, bend left] & 0 \end{tikzcd} \quad (*)$$

  is in bijection with

$$\mathbb{P}\left(H^{0,1}_{\bar{\partial}}(X, \mathrm{Hom}(Q, \mathcal{F}))\right)$$

- Call the extension sequence $(*)$ split if $[\beta] = 0$.

  iff $\exists$ an injection $Q \hookrightarrow \mathcal{E}$ lifting the proj $\mathcal{E} \to Q$.

- For such splitting $\mathcal{E} = \mathcal{F} \oplus Q$,

$$\bar{\partial}_{\mathcal{E}} = \begin{pmatrix} \bar{\partial}_F & \beta \\ 0 & \bar{\partial}_Q \end{pmatrix}$$

If $H = \begin{pmatrix} H_1 & \\ & H_2 \end{pmatrix}$ w.r.t $\mathcal{E} = \mathcal{F} \oplus Q$,
then the Chern connection ∇^H determined by $\bar{\partial}_E, H$ is

$$\nabla^H = \begin{pmatrix} \nabla^F & \beta \\ -\beta^{*h} & \nabla^Q \end{pmatrix},$$

where ∇^F, ∇^Q are the Chern connection

β^{*h} is the adjoint of β.

(1,0)-form (0,1)-form

The curvature of ∇^H is

$$F(\nabla^H) = \begin{pmatrix} F(\nabla^F) - \beta \wedge \beta^{*h} & \partial\beta \\ -\bar{\partial}\beta^{*h} & F(\nabla^Q) - \beta^{*h} \wedge \beta \end{pmatrix}.$$

§2. Moduli space of holo^{m} vector bundles

Key: To introduce stability on holo^{m} v.b.

Two motivations for stability:

① Original motivation due to Mumford is to provide the set of gauge equivalence classes of holo^{m} v.b with a "good" topology.

Mainly, the unstable ones cause "non-Hausdorff" problem.

② Turns out stability is an iff condition for a holo^{m} v.b admitting a soln to the Hermitian–Einstein eqn.

Let $\mu(E)$ denote the slope of E,
i.e. $\deg(E)/\operatorname{rank}(E)$.

Defn (Mumford)

- A holo^m v.b E is called stable semistable
if $\mu(F) \underset{\leq}{<} \mu(E)$ for any proper holo^m subbundle F of E.

- A holo^m v.b E is called polystable
if it is a direct sum of stable holo^m subbundles of the same slope.

Rmk: (1) Stability is preserved under gauge transformations.

(2) Stability is an open condition.

For an unstable vector bundle,

Prop. Given an arbitrary holo^{m} v.b,
$\exists!$ a Harder-Narasimhan filtration of $\mathcal{E}$,

$$0=\mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$$

s.t
- $\mathcal{E}_i/\mathcal{E}_{i-1}$ are semistable.
- $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ is strictly decreasing.

(Idea: Take $\mathcal{E}_1$ to be the maximal destabilizing subbundle of $\mathcal{E}$.)

For a semistable v.b,

Prop. Given a semistable v.b,
$\exists$ a Jordan - Hölder filtration of $\mathcal{E}$,

$$0=\mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots\cdots \subset \mathcal{E}_\ell = \mathcal{E}$$

s.t the quotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ are stable

Obviously, $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ are the same.

Denote by $Gr(E) = \bigoplus_{i=1}^{l} \mathcal{E}_i/\mathcal{E}_{i-1}$,

(graded v.b of E)

it is polystable.

<u>Defn</u>. Two semistable v.b are S-equiv if their graded v.b's are gauge equiv.

Rmk: $\{\text{stable}\} \subset \{\text{polystable}\} \subset \{\text{semistable}\}$.

<u>Denote</u>. $$\mathcal{M}^s(r,d) := \left\{ \begin{array}{l} \text{stable holo}^m \text{ v.b of} \\ \text{rk } r, \text{ deg } d \end{array} \right\} \Big/ \mathcal{G}$$

$$\mathcal{M}(r,d) := \left\{ \begin{array}{l} \text{polystable holo}^m \text{ v.b of} \\ \text{rk } r, \text{ deg } d \end{array} \right\} \Big/ \mathcal{G}$$

$$\cong \left\{ \begin{array}{l} \text{semistable holo}^m \text{ v.b of} \\ \text{rk } r, \text{ deg } d \end{array} \right\} \Big/ S\text{-equiv}$$

Note that when $(r,d)=1$, $\mathcal{M}^s(r,d) = \mathcal{M}(r,d)$.

And $M^s(r,d)$ is a smooth cpt complex mfld.

Ex 1. On $\mathbb{P}^1$, by Grothendick's thm, any holo^m v.b is of the form

$$E = \bigoplus_{i=1}^{k} \mathcal{O}(n_i).$$

Then $$\mu(E) = \sum_{i=1}^{k} n_i \Big/ K.$$

E is unstable unless all n_i's are equal.

polystable if all n_i's are equal.

stable only if $k=1$.

Ex 2. Consider the extension sequence

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(p) \to 0.$$

The isomorphism classes of $\mathcal{E}$ are parametrized by $\mathbb{P}(H^{0,1}_{\bar\partial}(X, \mathcal{O}(-p)))$

$$= \mathbb{P}\left(H^0(X, K(p))^*\right),$$

has $\dim_{\mathbb{C}} = g$.

Claim: Any non-split extension of this type is stable.

Pf: If $\mathcal{L} \hookrightarrow \mathcal{E}$ is a destabilizing line subbundle,

$$\left(\deg \mathcal{L} > \frac{\deg(\mathcal{E})}{rk(\mathcal{E})} = \frac{1}{2}. \right)$$

then $\deg \mathcal{L} \geq 1$.

Can compose $\mathcal{L} \hookrightarrow \mathcal{E} \to \mathcal{O}(p)$, obtain a holo^m map $\mathcal{L} \to \mathcal{O}(p)$, which is either 0 or an isomorphism.

(i) If $\mathcal{L} \to \mathcal{O}(p)$ is 0,

then $\mathcal{L} = \ker(\mathcal{L} \to \mathcal{O}(p))$

$$\subset \underset{\mathcal{O}.}{\underset{\parallel}{\mathrm{Ker}(\mathcal{E} \to \mathcal{O}(p))}} \quad \nearrow\!\!\!\!\swarrow.$$

(ii) If $\mathcal{L} \to \mathcal{O}(p)$ is an isom,

then $\mathcal{O}(p) \to \mathcal{L} \hookrightarrow \mathcal{E}$ is a nontrivial map

lifting the projection $\mathcal{E} \to \mathcal{O}(p)$.

So the extension splits. $\nearrow\!\!\!\!\swarrow$ ▨

Note that $\mathcal{E} = \mathcal{O} \oplus \underline{\underline{\underline{\mathcal{O}(p)}}}$ is unstable.

<u>Ex 3</u>: <u>Claim</u>: $0 \to \mathcal{O}_1 \to \mathcal{E} \to \mathcal{O}_2 \to 0$

with non-split extension

is strictly semistable.

Pf: • Not stable, since $\mathcal{O}_1$ has the same slope 0 as $\mathcal{E}$.

- semistable:

For any $\mathcal{L} \subset E$ a holo^{m} line subbundle,
The induced map $\mathcal{L} \to \mathcal{O}_2$ is either zero or nontrivial.

(i) If $\mathcal{L} \to \mathcal{O}_2$ is 0, then $\mathcal{L} \hookrightarrow \mathcal{O}_1$.
$\Rightarrow \deg \mathcal{L} \leq 0$.

(ii) If $\mathcal{L} \to \mathcal{O}_2$ is nontrivial, $\Rightarrow \deg \mathcal{L} \leq 0$.
▨

Note that $Gr(\mathcal{E}) = \mathcal{O}_1 \oplus \mathcal{O}_2$.

Claim: $\mathcal{O}_1 \oplus \mathcal{O}_2$ is contained in
holo^{m} direct sum
the closure of the gauge orbit of $\mathcal{E}$ with non-split extension.

Pf: Take the 1-parameter subgrp
$$g_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad (t>0).$$
Then
$$g_t^{*} \bar{\partial}_{\mathcal{E}} = g_t^{*}\left(\bar{\partial} + \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}\right))$$

$$= \bar{\partial} + \underset{\substack{\| \\ \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix}}}{g_t^{-1}} \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \underset{\substack{\| \\ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}.}}{g_t} + \underbrace{g_t^{-1} \bar{\partial} g_t}_{0}.$$

$$= \bar{\partial} + \begin{pmatrix} 0 & t^{-2}\beta \\ 0 & 0 \end{pmatrix}$$

$$= \bar{\partial} \quad \text{as } t \to \infty. \qquad \blacksquare$$

Let ω_X be a Kähler form on X to be normalized s.t $\int_X \omega_X = 1$.

Thm. (Narasimhan-Seshadri)

A holo^m v.b E carries a Herm metric h satisfying the Hermitian-Einstein equ

$$\underset{\substack{\uparrow \\ A^2(\mathrm{End}(E)).}}{F(\nabla^h)} = -2\pi i \cdot \mu(E) \cdot \mathrm{id}_E \cdot \omega_X$$

iff $\mathcal{E}$ is polystable.

Moreover, the soln h is unique up to multiplication by a positive constant if E is stable.

Rmk: From Chern-Weil theory,

$$\frac{i}{2\pi}\int_X \mathrm{Tr}(F(\nabla)) = \deg(E)$$

for any connection on E.

Use the eqn,

$$\frac{i}{2\pi}\int_X \mathrm{Tr}(F(\nabla^h)) = \frac{i}{2\pi}\int_X -2\pi i\cdot \mu(E)\cdot \mathrm{Tr}(\mathrm{id}_E)\cdot \omega_X$$

$$\|$$

$$\mu(E)\cdot \mathrm{rk}(E)\cdot \int_X \omega_X$$

$$\|$$

$$\deg(E).$$

<u>Rmk</u>: The first proof to N-S was algebraic and relates stable v.b with unitary reps of $\pi = \pi_1(X)$.
Donaldson presented an analytic proof.
N-S thm holds for cpt Kähler mflds of arbitrary dim, which is the Donaldson-Uhlenbeck-Yau thm.

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§3. Moduli space of Higgs bundles

Repeat all the defns to Higgs bundles.

<u>Defn</u>. A Higgs bundle (E,Φ) is stable
semistable
if $\mu(F) \underset{\leq}{<} \mu(E)$ for any proper ϕ-inv holo^{m} subbundle F of E.

polystable if $(E,\Phi) = \bigoplus_i (E_i, \Phi_i)$ stable w/ the same slope.

- H-N filtration for Higgs bundles
- J-H filtration for semistable Higgs bundle

  graded polystable Higgs bundles

  S-equiv

$$\mathcal{M}^{Higgs,s}(r,d) = \left\{ \begin{matrix} \text{stable Higgs bundles} \\ \text{of rk } r, \deg d \end{matrix} \right\} \Big/ G$$

$$\mathcal{M}^{Higgs}(r,d) = \left\{ \begin{matrix} \text{polystable Higgs bundles} \\ \text{of rk } r, \deg d \end{matrix} \right\} \Big/ G$$

$$\cong \left\{ \begin{matrix} \text{semistable Higgs bundles} \\ \text{of rk } r, \deg d \end{matrix} \right\} \Big/ \text{S-equiv}$$

- $\{\text{stable}\} \subset \{\text{poly stable}\} \subset \{\text{semistable}\}$
- $(r,d)=1$, $\mathcal{M}^{Higgs}(r,d) = \mathcal{M}^{Higgs,s}(r,d)$.

For our intersects, we also focus on $SL(n,\mathbb{C})$-Higgs bundles.

Defn. An $SL(n,\mathbb{C})$-Higgs bundle is a Higgs bundle $(\mathcal{E},\phi)$

s.t.
- $\det(\mathcal{E}) \overset{\sim}{=} \mathcal{O}$
- $\mathrm{tr}(\phi) = 0$.

Rmk: When we construct moduli space of polystable $SL(n,\mathbb{C})$-Higgs bundles, the gauge transformation lies in $SL(n,\mathbb{C})$.

Rmk: One can even consider G-Higgs bundles for reductive Lie grps G.

e.g. $G = SO(n,\mathbb{C}), SL(n,\mathbb{R}), Sp(2n,\mathbb{R}), \ldots\ldots$ and associated stability.

Ex. Fix a holo^m line bundle $K^{\frac{1}{2}}$ of K
($(K^{\frac{1}{2}})^2 = K$.)

Let $\mathcal{E} = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ holo^m direct sum.

Claim: The Higgs bundle $(\mathcal{E}, \phi_q)$
with $\phi_q = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} : \mathcal{E} \to \mathcal{E} \otimes K$
is stable, where $q \in H^0(X, K^{\otimes 2})$.

Pf: (i) $q=0$ case. $\phi_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

The only ϕ_0-inv holo^m subbundle
is $K^{-\frac{1}{2}}$, which has deg $1-g < \underset{\deg(E)}{\overset{\shortparallel}{0}}$.

So it is stable.

(ii) Use the fact that stability is an open condition

the Higgs bundle

$$(\mathcal{E}, \Phi_{\varepsilon q} = \begin{pmatrix} 0 & \varepsilon q \\ 1 & 0 \end{pmatrix}) \text{ is stable.}$$

(iii) Use $g = \begin{pmatrix} \varepsilon^{\frac{1}{4}} & \\ & \varepsilon^{-\frac{1}{4}} \end{pmatrix}$

Then (ii) means

$$(\mathcal{E}, g^{-1}\Phi_{\varepsilon q}\, g) \text{ is again stable}$$

$$\begin{aligned} &\| \\ &\underline{\begin{pmatrix} \varepsilon^{-\frac{1}{4}} & \\ & \varepsilon^{\frac{1}{4}} \end{pmatrix}} \begin{pmatrix} 0 & \varepsilon q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon^{\frac{1}{4}} & \\ & \varepsilon^{-\frac{1}{4}} \end{pmatrix} \\ &\| \\ &\varepsilon^{\frac{1}{2}} \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Claim: If (E, ϕ) is stable, so is $\underset{\forall t \in \mathbb{C}^*}{}(E, t\phi)$

Pf: Obvious. ▨

§4. Hitchin-Kobayashi correspondence

<u>Defn</u>. Given a Higgs bundle $(\mathcal{E}, \Phi)$ over X, call a Herm metric h **harmonic** if it solves the Hitchin eqn (Hitchin's self-duality eqn)

$$F(\nabla^h) + [\phi \wedge \phi^{*h}] = -2\pi i \cdot \mu(E) \cdot \mathrm{id}_E \cdot \omega_X$$

where
- ∇^h is the Chern connection.
- $F(\nabla^h)$ is the curvature.
- ϕ^{*h} is the adjoint of ϕ w.r.t h,

  i.e. $h(\phi s, t) = h(s, \phi^{*h} t)$.

Rmk: (i) Locally, if $\Phi = \varphi \, dz$ w.r.t some frame of E.

The metric presentation is h locally.

Then $\Phi^{*h} = \varphi^{*h} d\bar{z}$

$(\varphi^t h = h \overline{\varphi^{*h}} \Rightarrow \varphi^{*h} = \bar{h}^{-1} \bar{\varphi}^t \bar{h}.)$

Thus $[\Phi, \Phi^{*h}] = [\varphi, \varphi^{*h}] \, dz \wedge d\bar{z}$

Or globally, $[\Phi, \Phi^{*h}] = \phi \wedge \phi^{*h} + \phi^{*h} \wedge \phi.$

(ii) When $\deg(E) = 0$ ($\mu(E) = 0$),

Claim: the Hitchin eqn is equivalent to

$D = \nabla^h + \phi + \phi^{*h}$ is flat.

Then we obtain a map from

$$\{\text{Higgs bundles which admits harmonic metric}\}/G$$

$$\longrightarrow \{\text{flat connections}\}/G.$$

Pf of Claim:

$$F(D) = \underbrace{F(\nabla^h) + [\phi, \phi^{*h}]}_{} \text{ skew-Her} + \underbrace{\nabla^h(\phi + \phi^{*h})}_{} \text{ Herm} = 0$$

$$\Longleftrightarrow \begin{cases} F(\nabla^h) + [\phi, \phi^{*h}] = 0 \\ \nabla^h(\phi + \phi^{*h}) = 0. \end{cases}$$

$\Longleftrightarrow \Big\{$ Next time !