Higgs bundles and related topics.

Qiongling Li

Chern Institute of Mathematics,
Nankai University

qiongling.li@nankai.edu.cn

Contents:

Lecture 1:

Plan of the course

Part I: Basics of Higgs bundles
geometry of moduli space
NAH
Higher Teichmüller theory
parabolic Higgs bundles — 6

Part II: topics. — 6.

Today: Explain Betti, de Rham, Dolbeault moduli spaces.
Take a close look at rk 1 case.

Reference:
W. Goldman and E.Z. Xia,
"Rank one Higgs bundles and representations of fundamental grps of R.S"

§0. Equivalence of deformation theories.

Defn. A deformation theory (or transformation groupoid) (S, G) consisting of a category $\mathcal{C}$ defined by a grp action as follows:

Let $\alpha: G \times S \to S$ left action.

(S, G) consists of the category $\mathcal{C}$ with $\mathrm{Obj}(\mathcal{C}) = S$ with morphism $x \xrightarrow{g} y$ corresponding to the triple $(g, x, y) \in G \times S \times S$ s.t $a(g,x) = y$.

- $e \in G$ determines the identity morphism $x \xrightarrow{e} x$.
- $x \xrightarrow{g} y$ has an inverse $y \xrightarrow{g^{-1}} x$
- composition.

Defn. The moduli set corresponding to such a groupoid is the set $Iso(\mathcal{L})$ of isomorphism classes of objects.

Defn. An equivalence of categories is a functor $F: \mathcal{A} \to \mathcal{B}$ s.t $\exists H: \mathcal{B} \to \mathcal{A}$ and $F \circ H \cong I_{\mathcal{B}}$

$H \circ F \cong I_{\mathcal{A}}$.

$\rightsquigarrow$ a bijection : $Isom(\mathcal{A}) \to Isom(\mathcal{B})$.

Prop. (Criterion) A functor $F: \mathcal{A} \to \mathcal{B}$ is an equiv iff

(1) surjective on Isomorphism classes.

(2) Full : $F(x,y): Mor(x,y) \to Mor(F(x), F(y))$ is surjective.

(3) Faithful : injective.

§1. $\boxed{\text{The Betti groupoid.}}$

Fix G a structure grp, e.g. $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $U(n)$;

Σ a compact smooth oriented surface with fundamental grp π.

- The objects are representations : $\pi \to G$

$$S = Hom(\pi, G)$$

- The morphisms are from G by conjugation.

$$G \times Hom(\pi, G) \longrightarrow Hom(\pi, G)$$

$$g \cdot \rho \longmapsto g^{-1} \rho g$$

Defn. The Betti groupoid is $(\mathrm{Hom}(\pi, G), G)$.

- π admits a presentation
$$\langle A_1, B_1, \dots, A_g, B_g \mid [A_1, B_1] \cdots [A_g, B_g] = 1 \rangle$$

The map
$$\mathrm{Hom}(\pi, G) \hookrightarrow G^{2g}$$
$$\rho \longmapsto (\rho(A_1), \rho(B_1), \dots, \rho(A_g), \rho(B_g)).$$
embeds $\mathrm{Hom}(\pi, G)$ as a Zariski-closed subset of G^{2g} defined $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1$. $(*)$;

- If G is abelian, it acts trivially on $\mathrm{Hom}(\pi, G)$.
The condition $(*)$ is automatically satisfied.
So $\mathrm{Hom}(\pi, G)/G \cong \mathrm{Hom}(\pi, G) \cong G^{2g}$.
$$\begin{array}{c} \mathrm{Hom}(\pi, G)/G \\ \| \\ \mathrm{Isom}((\mathrm{Hom}(\pi, G), G)). \end{array}$$
will apply this to $G = \mathbb{C}^*, U(1), \mathbb{R}^+$.

§2. $\boxed{\text{The de Rham groupoid}}$

Let E be a smooth complex vector bundle over Σ.
$\mathcal{A}^k(\Sigma)$ denote the space of k-forms on Σ
$\mathcal{A}^k(\Sigma, E)$ - - - - - - E-valued k-forms.

Defn. A gauge transformation of E is a smooth bundle automorphism
$$\begin{array}{ccc} \xi : E & \longrightarrow & E \\ \downarrow & \circlearrowright & \downarrow \\ \mathrm{id} : \Sigma & \longrightarrow & \Sigma \end{array}$$

Denote by $\mathcal{G}(E)$ the group of gauge transformations of E.

Defn. (Connection)

A connection on E is an operator
$$D : \mathcal{A}^0(\Sigma; E) \longrightarrow \mathcal{A}^1(\Sigma, E)$$
s.t $\quad D(fs) = f D(s) + df \wedge Ds.$

Such a map extends to $D: \mathcal{A}^p(\Sigma; E) \to \mathcal{A}^{p+1}(\Sigma; E).$

Denote by $\mathcal{U}(E)$ the space of all connections on E.

Note that fix a connection D_0, an arbitrary connection
$$D = D_0 + \eta \quad \text{for } \eta \in \mathcal{A}^1(\Sigma; \mathrm{End}(E)).$$

So $\mathcal{U}(E)$ is an affine space modeled on $\mathcal{A}^1(\Sigma; \mathrm{End}(E))$.

Defn. (Curvature)

The curvature of a connection D is defined as $\quad F(D)\, s = D \circ D(s)$,

turns out to be an $\mathrm{End}(E)$-valued 2-form
$$F(D) \in \mathcal{A}^2(\Sigma; \mathrm{End}(E)).$$

Call D flat if $F(D) = 0$.

Denote by $\mathcal{F}(E)$ the space of flat connections on E.

(Note that for the existence of a flat connection, require $\deg(E) = 0$.)

- <u>The gauge action on connections</u>

$\xi^* D$ is defined as $\quad (\xi^* D)(s) = \xi^{-1} D(\xi s)$

for $\xi \in \mathcal{G}(E)$.

$$\xi \cdot D := (\xi^{-1})^* D.$$

Locally, w.r.t a frame e,

$$D = d + \eta \quad (\text{i.e } De = e\eta)$$

Then $\xi^* D = d + g^{-1}\eta g + g^{-1} dg$

(Here, g is the local expression of ξ w.r.t e.)

i.e. $\xi e = eg$.

$$(\xi^* D)(e) = D(eg) = e(\eta g + dg)$$
$$= eg(g^{-1}\eta g + g^{-1} dg).$$

- $F(\xi^* D) = \xi^*(F(D))$.

Hence, $\mathcal{G}(E)$ preserves flatness.

Defn. The de Rham groupoid is $(\mathcal{F}(E), \mathcal{G}(E))$.

§3. Equivalence between Betti and de Rham groupoids

Start from a flat connection D on a vector bundle E, want to obtain a rep $\rho: \pi \to GL(n, \mathbb{C})$.

Locally, w.r.t a frame e, $De = e \cdot \eta$.

Over a smooth path $\gamma: [0,1] \to \Sigma$, parallel transport defines a linear map between the fibers $P_{\gamma(t)}: E_{\gamma(0)} \longrightarrow E_{\gamma(t)}$.

That is, $P_{\gamma(t)}(v)$ is parallel w.r.t D, for $v \in E_{\gamma(0)}$.

Suppose $v = (e\circ\gamma(0))\cdot f(0) \in E_{\gamma(0)}$.

Then $P_{\gamma(t)}(v) = (e\circ\gamma(t))\cdot \underline{g(t)}\cdot f(0)$ is parallel to D

$$\iff D_{\partial/\partial t}\big((e\circ\gamma(t))\cdot g(t)\cdot f(0)\big) = 0$$

$$\iff (e\circ\gamma(t))\,\big(\underline{\eta\circ\gamma(t)\cdot g(t) + dg(t)}\big)(\partial/\partial t)\cdot f(0) = 0$$

$$\iff g'(t) + (\eta\circ\gamma(t))\cdot g(t) = 0$$

$$\iff g(t) = \exp\Big(-\int_0^t \gamma^*\eta\Big)$$

<u>Fact</u>: Flatness of D implies the parallel transport only depends on homotopic class of γ relative to its endpts.

Now we obtain a homomorphism: fix a pt $p \in E_{x_0}$.

$$\mathrm{hol}_p(D) : \pi_1(X, x_0) \longrightarrow GL(n,\mathbb{C})$$
$$\gamma \longmapsto \big(P_\gamma : \underset{\gamma(0)}{\overset{\shortparallel}{E_{x_0}}} \to \underset{\gamma(1)}{\overset{\shortparallel}{E_{x_0}}}\big)^{-1}$$

w.r.t a fixed frame e at E_{x_0}.

<u>Thm</u>: The holonomy functor
$$\mathrm{hol}_p : (\mathcal{F}(E), \mathcal{G}(E)) \longrightarrow (\mathrm{Hom}(\pi, GL(n,\mathbb{C})), GL(n,\mathbb{C}))$$
is an equivalence of groupoids.

Pf: • surjective on isomorphism classes.

Given a rep $\rho \in \mathrm{Hom}(\pi, GL(n,\mathbb{C}))$, we construct a flat vector bundle $\mathbb{C}_\rho \to \Sigma$ as follows:

the grp π acts on the total space $\check{\Sigma} \times \mathbb{C}^n$ by

$$\gamma \cdot (\check{s}, x) := (\underset{\substack{\uparrow \\ \text{deck transformation.}}}{\gamma \cdot \check{s}}, \rho(\gamma)x) \quad \forall \gamma \in \pi.$$

"$\check{s} \cdot \gamma^{-1}$"

The quotient $(\check{\Sigma} \times \mathbb{C}^n)/\pi$ is the total space of a smooth vector bundle $\mathbb{C}_\rho \xrightarrow{P} \Sigma$, which carries a natural flat connection D as the descending of $D_0 = d$ on $\check{\Sigma} \times \mathbb{C}^n$.

$[(\check{\Sigma}, \underset{\substack{\uparrow \\ \text{constant}}}{v})]$ is parallel to D.

So this D gives holonomy ρ up to conjugation. ▨

- Full and faithful (need to check.)

§4. $\boxed{\text{Rank 1 case for equivalence between Betti and de Rham moduli spaces.}}$

Let E be a trivial complex line bundle over Σ.

A trivialization τ is a global frame of E.

- The gauge transformation $\xi \in \mathcal{G}(E)$ is determined by a smooth map $g : \Sigma \to \mathbb{C}^*$ via

$$\xi(\tau) = g \cdot \tau.$$

$$\mathcal{G}(E) \cong \mathrm{Map}(\Sigma, \mathbb{C}^*).$$

The subgrp $\mathcal{G}_u(E) \cong \mathrm{Map}(\Sigma, U(1))$

Let $\underset{\mathcal{G}(E)^\circ =}{\mathrm{Map}(\Sigma, \mathbb{C}^*)^\circ}$ denote the component containing the constant map.

$$G(E)/G(E)^\circ = \pi_0(G(E))$$

Note that $\quad \mathrm{Map}(\Sigma, \mathbb{C}^*)^\circ \cong \mathcal{A}^0(\Sigma)$

$$g \longmapsto \log g.$$

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ \Sigma \arrow[r, "g"'] \arrow[ur, dashed, "\log g"] & \mathbb{C}^* \end{tikzcd}$$ iff $g_*: \pi_1\Sigma \to \pi_1(\mathbb{C}^*)$ is trivial

So $\quad \mathcal{F}(E)/G(E) = \underline{\underline{\left(\mathcal{F}(E)/G(E)^\circ\right)}} \Big/ \underline{\underline{\pi_0(G(E))}}.$

- On E, there is a unique connection D_0 s.t. $D_0\tau = 0$.

  Any connection D is of the form

  $$D = D_0 + \eta, \quad \eta \in \mathcal{A}^1(\Sigma).$$

  D is flat $\Longleftrightarrow d\eta = 0$.

  $\xi^*(D_0 + \eta) = D_0 + \eta + g^{-1}dg. \quad (\xi \leftrightarrow g \in \mathrm{Map}(\Sigma, \mathbb{C}^*)$

  If $g \in \mathrm{Map}(\Sigma, \mathbb{C}^*)^\circ$, $\; g^{-1}dg = d\log g$.

  So $\quad \mathcal{F}(E)/G(E)^\circ \cong Z^1(\Sigma)/B^1(\Sigma) = H^1(\Sigma).$

The Betti moduli space is $\mathrm{Hom}(\pi, \mathbb{C}^*) \cong \mathrm{Hom}(\pi, S^1) \times \mathrm{Hom}(\pi, \mathbb{R}^+)$

The de Rham moduli space:

- $$\begin{array}{ccccc} \mathcal{F}(E) & = & \mathcal{F}_u(E) & \times & \mathcal{A}^1(\Sigma, \mathbb{R}) \\ D_0 + \eta & & D_0 + i\,\mathrm{Im}\,\eta & & \mathrm{Re}\,\eta \end{array}$$

- $\mathcal{G}(E) \cong \mathrm{Map}(\Sigma, \mathbb{C}^*) = \underbrace{\mathrm{Map}(\Sigma, S^1)}_{\mathcal{G}_u(E)} \times \mathrm{Map}(\Sigma, \mathbb{R}^+)$

$g \mapsto (g_u, g_r)$

$$g^*(D_0+\eta) = D_0 + \eta + g^{-1}dg$$
$$= \left(D_0 + i\,\mathrm{Im}\,\eta + \underline{g_u^{-1}dg_u}\right) + (\underbrace{g_r^{-1}dg_r}_{d\log g_r} + \mathrm{Re}\,\eta)$$

- $\mathcal{F}(E)/\mathcal{G}(E) \cong \mathcal{F}_u(E)/\mathcal{G}_u(E) \times H^1(\Sigma, \mathbb{R})$.

$$\cong \left(\underbrace{\mathcal{F}_u(E)/\mathcal{G}_u^0(E)}_{H^1(\Sigma, i\mathbb{R})}\right)\Big/\pi_0(\mathcal{G}_u(E)) \times H^1(\Sigma, \mathbb{R})$$

(will see $\cong H^1(\Sigma, i\mathbb{R})/H^1(\Sigma, \mathbb{Z}) \times H^1(\Sigma, \mathbb{R})$.)

The equivalence between the moduli spaces is given by

$$\mathrm{hol}_p : \mathcal{F}(E) \longrightarrow \mathrm{Hom}(\pi, \mathbb{C}^*)$$
$$D_0 + \eta \longmapsto \left(\gamma \mapsto \exp\left(\int_\gamma \eta\right)\right)$$

Restrict to $\mathrm{hol}_p : \mathcal{F}_u(E) \longrightarrow \mathrm{Hom}(\pi, U(1))$

<u>Claim:</u> $\ker(\mathrm{hol}_p) = \mathcal{G}(E)$.

Assuming the claim: hol_p desends to a map

$$\rho : \left(\underbrace{\mathcal{F}_u(E)/\mathcal{G}_u(E)}_{H^1(X, i\mathbb{R})}\right)\Big/\pi_0(\mathcal{G}_u(E)) \longrightarrow \mathrm{Hom}(\pi, U(1)) = \mathrm{Hom}(\pi, i\mathbb{R})\big/\mathrm{Hom}(\pi, 2\pi i\mathbb{Z})$$

$$\Rightarrow \pi_0(G_{\mathcal{U}}(E)) \underset{\sim}{=} H^1(\Sigma, \mathbb{Z}).$$

$$\overset{\shortparallel}{\mathbb{Z}\langle w_1, \cdots, w_{2g}\rangle}$$

where $w_1, \cdots, w_{2g}$ has period $\in 2\pi i\mathbb{Z}$ and form a basis.

<u>Pf of Claim:</u>

"$\subset$" If $\eta \in H^1(\Sigma, i\mathbb{R})$ s.t $\exp(\int_\gamma \eta) = 1 \quad \forall \gamma \in \pi$.

define $g(p) = \exp \int_{x_0}^{p} \eta$ well-defined.

as a fcn $g: \Sigma \to \mathbb{C}^*$.

and $\eta = g^{-1} dg$.

"$\supset$" Given $g^{-1}dg$ for $g: \Sigma \to \mathbb{C}^*$,

one can lift g to $\tilde{g}: \tilde{\Sigma} \to \mathbb{C}^*$.

then $\tilde{g}^{-1} d\tilde{g}$ is exact on $\tilde{\Sigma}$.

$$\overset{\shortparallel}{d \log \tilde{g}}.$$

Take a lift $\tilde{\gamma}$ of γ to $\tilde{\Sigma}$,

then
$$\int_\gamma g^{-1}dg = \int_{\tilde{\gamma}(0)}^{\tilde{\gamma}(1)} \tilde{g}^{-1} d\tilde{g} = \int_{\tilde{\gamma}(0)}^{\tilde{\gamma}(1)} d\log\tilde{g}$$
$$= \log\tilde{g}(\tilde{\gamma}(1)) - \log\tilde{g}(\tilde{\gamma}(0))$$
$$\in 2\pi i \mathbb{Z}.$$

$$\Rightarrow \exp \int_\gamma g^{-1} dg = 1. \qquad \blacksquare$$

Lecture 2

§5. The Dolbeault groupoid

Let X be a Riemann surface diffeo to Σ.

$$\mathcal{A}^1(X) = \underset{dz}{\mathcal{A}^{1,0}(X)} \oplus \underset{d\bar{z}}{\mathcal{A}^{0,1}(X)}$$

Hodge $*$-operator on $\mathcal{A}^1(X)$

$$*dz = -i\,dz$$
$$*d\bar{z} = i\,d\bar{z}.$$

__Defn__. Given a complex vector bundle E over X, a holo^m str on E is a diff operator

$$\bar{\partial}_E : \mathcal{A}^{p,q}(X,E) \to \mathcal{A}^{p,q+1}(X,E)$$

which satisfy

$$\bar{\partial}_E(f \cdot s) = \bar{\partial} f \wedge s + f \cdot \bar{\partial}_E s.$$
$$\forall f \in \mathcal{A}^0(X),\ s \in \mathcal{A}^{p,q}(X,E).$$

Rmk: If we are dealing w/ higher dim base mfld, we add the integrability condition $\bar{\partial}_E^2 = 0$.

Defn. A Higgs bundle over X of rk n is a pair (E, Φ) where

- E is a holo^m v.b over X of rk n.
- Φ is a holo^m 1-form on X taking values in $\underbrace{\mathrm{End}(E)}_{\text{a holo}^m\text{ v.b.}}$.

i.e. Φ is a holo^m section of

$$\underbrace{T^*X}_{K} \otimes \mathrm{End}(E)$$

holo^m str induced by X's complex str.

Φ is called Higgs field.

Rmk: If the base mfld is of higher dim, add the integrability condition $\Phi \wedge \Phi = 0$.

- The $\mathcal{G}(E)$-action on the space of Higgs bundles is as follows:

$$(\bar\partial_E, \Phi) \xrightarrow{g} (g^*\bar\partial_E, \; g^*\phi)$$

$$g^*\bar\partial_E = g^{-1}\bar\partial_E(g\,\bullet), \qquad g^*\phi = g^{-1}\phi g.$$

Locally, w.r.t a frame e,

$$\bar\partial_E e = e\Phi \qquad (\bar\partial_E = \bar\partial + \Phi).$$

$$(g^*\bar\partial_E)e = e(g^{-1}\Phi g + g^{-1}\bar\partial g).$$

$$g \Leftrightarrow g.$$

$$(g^*\phi)e = e\cdot g^{-1}\phi g.$$

<u>Defn</u>. The Dolbeault groupoid is

$$(\mathrm{Higgs}(E), \; \mathcal{G}(E))$$

$\mathrm{Higgs}(E)$: space of Higgs bundles over X.

§5.1 Understand the Dolbeault groupoid in rk 1, deg 0 case.

E is trivial because rk 1, deg 0 condition.

Start with a trivial complex line bundle E.

A Higgs field on the hol^m line bundle $(E, \bar{\partial}_E)$ is just a hol^m 1-form on X.

since $End(E) = E \otimes E^* = \mathcal{O}$.

- Thus $\underset{\substack{\Uparrow \\ \text{space of hol}^m\text{ str's on } E}}{\mathrm{Higgs}(E) = \mathrm{Hol}(E)} \times \underline{\underline{H^{1,0}(X)}}.$

There is a standard base pt in $\mathrm{Hol}(E)$.

$\bar{\partial}_0 = \bar{\partial}$ for $X \times \mathbb{C}$.

An arbitrary hol^m str on E is of the form $\bar{\partial}_0 + \Phi$, where $\Phi \in \mathcal{A}^{0,1}(X)$.

- The gauge action $\mathcal{G}(E)$ on $\mathrm{Hol}(E)$

$$\bar\partial_0 + \Phi \longmapsto \bar\partial_0 + \Phi + \underbrace{g^{-1}\bar\partial g}.$$

Again, $\mathcal{G}(E) / \mathcal{G}(E)^0 = \pi_0(\mathcal{G}(E))$

$$\mathrm{Hol}(E) / \mathcal{G}(E) = \left(\mathrm{Hol}(E) / \mathcal{G}(E)^0 \right) / \pi_0(\mathcal{G}(E))$$

$\mathcal{G}(E) = \mathrm{Map}(X, \mathbb{C}^*)$

$\mathcal{G}(E)^0 = \mathrm{Map}(X, \mathbb{C}^*)^0$ (containing constant maps)

$g \in$ $\quad g = \exp(f)$.

So $\mathrm{Hol}(E) / \mathcal{G}(E)^0 \cong \mathcal{A}^{0,1}(X) /$ exact (0,1)-form $\bar\partial \mathcal{A}^0(X)$.

By the Hodge decomposition

$$\mathrm{Hol}(E)/\mathcal{G}(E)^0 \cong H^{0,1}(X).$$

So the Dolbeault moduli space

$$\mathrm{Higgs}(E)/\mathcal{G}(E) \cong \mathrm{Hol}(E)/\mathcal{G}(E) \times H^{1,0}(X)$$
$$\cong \underbrace{H^{0,1}(X)/\pi_0(\mathcal{G}(E))}_{} \times H^{1,0}(X).$$

Claim: $\pi_0(\mathcal{G}(E))$'s image form a lattice of rk $2g$ in $H^{0,1}(X)$.

(From last time, $\pi_0(\mathcal{G}(E))$'s image in $H^1(X)$ is a lattice of rk $2g$.)

$\cong \operatorname{Jac}(X) \times H^{1,0}(X)$.

$\Uparrow$ a complex torus of dim g.

- Identify $\operatorname{Higgs}(E)/\mathcal{G}(E)$ with $T^*\operatorname{Jac}(X)$.

Consider the Hermitian form on $A^1(X)$ by $\langle \alpha, \beta \rangle := \int_X \alpha \wedge * \bar{\beta}$.

$\begin{pmatrix} \text{pos. def on } A^{0,1}(X) \\ \text{neg. def on } A^{1,0}(X). \end{pmatrix}$ $dz \wedge -i d\bar{z}$.

Its restriction to $V = H^{0,1}(X)$ defines an isomorphism $\bar{V} \longrightarrow V^*$. of complex v.s.

$$\operatorname{Higgs}(E)/\mathcal{G}(E) = \underset{\underset{\operatorname{Jac}(X)}{\cong}}{H^{0,1}(X)/L} \times \underbrace{\overset{\cup}{H^{1,0}(X)}}_{\underset{\bar{V}}{\cong}}$$

The tangent space of $\mathrm{Jac}(X)$ at any pt identifies with V.

Thus $V/L \times \bar{V} \cong V/L \times V^*$

$\Rightarrow \mathrm{Higgs}(E)/\mathcal{G}(E) \cong T^*\mathrm{Jac}(X)$.

---

§6. Equivalence between the de Rham and Dolbeault groupoids for rk 1, deg 0 case.

§6.1. Introduce Hermitian metrics.

<u>Defn</u>. A Herm metric H on E is a smooth family of pos. def Herm forms $\langle , \rangle_H : E_x \times E_x \to \mathbb{C}$.

Denote by $Her(E)$ the space of Herm metrics on E.

In terms of a basis / frame e,

$$H(\underline{e \cdot \xi}, \underline{e \cdot \eta}) = \xi^t h \eta$$

where $h = H(e, e)$ Hermitian matrix.

- The action of $\mathcal{G}(E)$ on $Her(E)$, locally, $g \cdot h = (g^{-1})^t h g^{-1}$.
- If E is a flat vector bundle on X with holonomy $\phi: \pi \to GL(r, \mathbb{C})$, then a Hermitian metric $H \in Her(E)$ corresponds to a ϕ-equivariant map

$$h : \tilde{X} \longrightarrow Her(\mathbb{C}^r)$$

(i.e. $h(\gamma \cdot x) = \phi(\gamma) h(x)$.)

Idea: Parallel transport H along paths based at x_0 to E_{x_0} w.r.t this flat connection.

i.e. $h([\gamma])^{(u,u)} = H(s(\gamma(1)), s(\gamma(1)))$

where $s(\gamma(t))$ is a parallel section along γ starting from u.

- Induced Hermitian pairing over $\mathcal{A}^*(X, E)$

$$\mathcal{A}^k(X, E) \times \mathcal{A}^l(X, E) \to \mathcal{A}^{k+l}(X).$$

Defn. A connection D is unitary w.r.t H if

$$d\langle s_1, s_2\rangle_H = \langle Ds_1, s_2\rangle_H + \langle s_1, Ds_2\rangle_H.$$

Prop. Given $(E, \bar\partial_E)$ with H,

$\exists!$ a connection D s.t

(1) $D^{0,1} = \bar\partial_E$.

(2) D is unitary w.r.t H.

D is called Chern connection.

Prop. Given a connection D and H,

$\exists!$ a decomposition

$$D = \underbrace{D_H}_{\text{unitary connection w.r.t } H} + \underbrace{\Phi_H}_{\text{self-adjoint w.r.t } H}$$

s.t

$$\left(H(\Phi_H s, t) := \frac{1}{2}\{H(Ds, t) + H(s, D(t)) - d(H(s,t))\}. \right)$$

§6.2. Restrict to the case
E is a complex trivial line bundle
with a $\underbrace{\text{frame } \tau}_{\text{trivialization}}$.

Let H_0 be $H_0(\tau, \tau) = 1$.

$G(E) = Map(X, \mathbb{C}^*) \ni g$.

- $G(E)$ acts on $Her(E)$ as

$$h \longmapsto |g|^{-2} h.$$

$$\left(\langle u, v \rangle_{g \cdot H} := \langle g^{-1}u, g^{-1}v \rangle_H \right)$$

$$h = H(\tau, \tau) : X \to \mathbb{R}^+.$$

- $G(E)$ acts on $Her(E)$ transitively.

Want $g \cdot h_1 = h_2$,

need $g(z) = \sqrt{h_1(z)/h_2(z)}$.

- $D = D_0 + \eta$ is unitary w.r.t H

  iff

  $$d(H(\tau,\tau)) = H(D\tau,\tau) + H(\tau, D\tau)$$

  $$\Rightarrow \quad dh = h\eta + h\bar{\eta}$$

  $$\Rightarrow \quad h^{-1}dh = \eta + \bar{\eta} = 2\mathrm{Re}(\eta).$$

- Φ is self-adjoint w.r.t H

  iff $\Phi = \bar{\Phi}$ i.e. Φ is real.

  $$\left(H(\Phi\tau,\tau) = H(\tau,\Phi\tau) \right)$$

- w.r.t H, $D = D_0 + \eta$ is uniquely decomposed $D = D_H + \Phi_H$,

  $$\begin{cases} D_H = D_0 + i\,\mathrm{Im}\,\eta + \frac{1}{2}h^{-1}dh. \\ \Phi_H = \mathrm{Re}\,\eta - \frac{1}{2}h^{-1}dh. \end{cases}$$

§6.2(a). $\boxed{\text{Start from a flat connection}}$

Goal: To find a decomposition and further obtain a Higgs bundle.

Idea: Find the "best" H and use H to decompose.

<u>Defn</u>. A Hermitian metric h is <u>harmonic</u> w.r.t $D \in \mathcal{F}(E)$ if the associated equivariant map
$$\tilde{h} : \tilde{X} \to \mathrm{Herm}(\mathbb{C}) = \mathbb{R}^+.$$
(*) corresponding to h

is a <u>multiplicatively harmonic fcn</u>.

(defined as, its logarithm is a harmonic fcn.)

Prop. Condition (*) holds
iff $\tau \Sigma_H$ is a harmonic 1-form.
(iff $(\tau \Sigma_H)^{1,0}$ is holo^m.)

Pf: For a path $\gamma: [0,1] \to X$ with $\gamma(0) = x_0$.
Let s be the flat section from
τ along γ, then
$(s(x_0) = \tau)$

$$s(\gamma(1)) = \exp\left(-\int_\gamma \eta\right) \cdot \tau_0$$

for $D = D_0 + \eta$.

Use the defn of $\check{h}$,

$$\check{h}([\gamma]) := H(s(\gamma(1)), s(\gamma(1)))$$

$$\underset{\tilde{X}}{\uparrow} \qquad = H\left(\exp\left(-\int_\gamma \eta\right) \cdot \tau_0, \exp\left(-\int_\gamma \eta\right) \cdot \tau_0\right)$$

$$= \exp\left(-\int_\gamma (\eta + \bar{\eta})\right) \cdot \frac{\underbrace{H(\tau_0, \tau_0)}}{h}$$

$$= \exp\left(-2\int_\gamma \mathrm{Re}(\eta) + h^{-1}dh\right)$$

$$= \exp\left(-2\int_\gamma \left(\mathrm{Re}(\eta) - \tfrac{1}{2}h^{-1}dh\right)\right)$$

$$= \exp\left(-2\int_\gamma \Psi_H\right). \quad \blacksquare$$

<u>Prop</u>. For $D \in \mathcal{F}(E)$, $\exists!$ a harmonic metric H (up to constant scalar).

Pf: $\overset{D_0+\eta}{D} = D_H + \Psi_H$

$$= (D_0 + i\,\mathrm{Im}\,\eta + \tfrac{1}{2}h^{-1}dh)$$

$$+ \underline{\underline{(\mathrm{Re}\,\eta - \tfrac{1}{2}h^{-1}dh)}}$$

One wants to find h s.t

$\mathrm{Re}\,\eta - \frac{1}{2} h^{-1} dh$ is harmonic.

By Hodge decomposition,

$$\mathrm{Re}\,\eta = \text{harmonic 1-form} + \underline{dS}$$

Let $h = e^{2S}$. $\square$

$$\begin{tikzcd}
D \arrow[r] & (D, H) \arrow[r] & (D_H, \Psi_H) \arrow[d] \\
 & & (D_H^{0,1}, \Psi_H^{1,0})
\end{tikzcd}$$

D: Flat H: harm Ψ_H: harm.

$\Psi_H^{1,0}$: holo^m

a Higgs bundle.

§6.2(b). $\boxed{\text{Start from a Higgs bundle}}$

One wants to find a way to sum up and obtain a flat connection.

Defn. A Hermitian metric is harmonic w.r.t $\bar{\partial}_E$ iff the Chern connection D^H is flat.

iff $D^H + \phi + \bar{\phi}$ is flat.

Prop. For each $\bar{\partial}_E \in \mathrm{Hol}(E)$, $\exists!$ $h \in \mathrm{Her}(E)$ up to constant s.t h is harmonic w.r.t $\bar{\partial}_E$.

Pf: Write $\bar{\partial}_E = \bar{\partial}_0 + \Psi$

$$= \bar{\partial}_0 + (\underset{\underset{\mathcal{H}^{0,1}(X)}{\cap}}{\Psi_0} + \bar{\partial} s)$$

Note that H_0 is harmonic w.r.t $\bar{\partial}_0 + \Psi_0$,

$(D^{H_0} = D_0 + \Psi_0 - \bar{\Psi}_0.)$

Then $g \cdot H_0$ is harmonic to $\underline{g \cdot (\bar{\partial}_0 + \Psi_0)}$.

Let $g = e^{s}$,

$$g^{-1}\cdot(\bar\partial_0 + \Phi_0) = \bar\partial_0 + \Phi_0 + \bar\partial s = \bar\partial_E$$

So $g^{-1}\cdot H_0$ is the desired metric. ∎

$\quad\;\;\shortparallel$
$e^{2s} H_0$

§6.2(c) $\boxed{\text{Combine}}$

Denote by $(\mathcal{F}(E)\times Her(E))_{harm}$ the subset of (D,H) s.t. H is harm w.r.t D

Denote by $(Higgs(E)\times Her(E))_{harm}$ the subset of $(\bar\partial_E, \phi, H)$ s.t. H is harm w.r.t $\bar\partial_E$.

We have a diagram:

$$
\begin{tikzcd}
(\mathcal{F}(E), \mathcal{G}(E)) \arrow[d, leftrightarrow] & & (\mathrm{Higgs}(E), \mathcal{G}(E)) \arrow[d, leftrightarrow] \\
((\mathcal{F}(E) \times \mathrm{Herm}(E))_{harm}, \mathcal{G}(E)) & & ((\mathrm{Higgs}(E) \times \mathrm{Herm}(E))_{harm}, \mathcal{G}(E))
\end{tikzcd}
$$

$$(D, H) \longmapsto (D_H^{0,1}, \Phi_H^{1,0}, H)$$

$$(D^H + \phi + \bar{\phi}, H) \longleftarrow\!\shortmid (\bar{\partial}_E, \phi, H)$$

<u>Thm</u>. The induced functor
$$(\mathcal{F}(E), \mathcal{G}(E)) \longrightarrow (\mathrm{Higgs}(E), \mathcal{G}(E))$$
is an equivalence of groupoids.

Pf: The rest need to check. ∎

---

§7. Complex structures on moduli spaces. (again rk 1 cases.)

- Betti moduli space

$$\mathcal{M}_{Betti} = \mathrm{Hom}(\pi, \mathbb{C}^*) \cong (\mathbb{C}^*)^{2g}$$

$$T_\rho \mathrm{Hom}(\pi, \mathbb{C}^*) \cong \mathrm{Hom}(\pi, \mathbb{C}) = \mathbb{C}^{2g}.$$

$$J_1 : T_\rho \mathcal{M}_{Betti} \longrightarrow T_\rho \mathcal{M}_{Betti}$$
$$X \longmapsto iX.$$

- De Rham moduli space.

$$\mathcal{M}_{de\,Rham} \cong H^1(X, i\mathbb{R}) \Big/ H^1(X, \mathbb{Z}) \times H^1(X, \mathbb{R})$$

$$\cong H^1(X) \Big/ H^1(X, \mathbb{Z})$$

$$T_\eta \mathcal{M}_{de\,Rham} \cong H^1(X)$$

$$J_2 : T_\eta \mathcal{M}_{de\,Rham} \longrightarrow T_\eta \mathcal{M}_{de\,Rham}$$
$$X \longmapsto iX.$$

In fact, the equivalence between

$$\mathcal{M}_{\text{de Rham}} \longrightarrow \mathcal{M}_{\text{Betti}}$$

$$D_0 + \eta \longmapsto \left(\gamma \mapsto \exp \int_\gamma \eta\right)$$

The tangent map is at η

$$X \longmapsto \left(\gamma \mapsto \exp \int_\gamma \eta \cdot \int_\gamma X\right)$$

is a biholo^m w.r.t J_1, J_2.

So we can say $J_1 = J_2$, denoted by J.

- Dolbeault moduli space : $\mathcal{M}_{Dol}$, M_{Higgs}

$$\mathcal{M}_{Dol} = \underbrace{H^{0,1}(X)/L}_{Jac(X)} \times H^{1,0}(X)$$

$$T_\sigma \mathcal{M}_{Dol} \cong H^{0,1}(X) \times H^{1,0}(X)$$

$$(\Psi, \Phi)$$

$$I : T_\sigma M_{Dol} \longrightarrow T_\sigma M_{Dol}$$

$$(\alpha, \Phi) \longmapsto (i\alpha, i\Phi).$$

I is different from J.

Lecture 3

X — a compact R.S
of $g \geq 2$ if not specified.

$\mathcal{E}$ — $(E, \bar{\partial}_E)$ holo^m v.b.

Goal today:

- moduli space of polystable vector bundles / Higgs bundles
- relate the stability with soln to Hitchin eqn.

§1. $\boxed{\text{Preparation: extensions of holo}^m \text{ vector bundles.}}$

If $\mathcal{F} \subset \mathcal{E}$ is a holo^m subbundle, then $Q = \mathcal{E}/\mathcal{F}$ has an induced holo^m str, $\bar{\partial}_Q$.

Moreover, we can choose a complement subbundle of $\mathcal{F}$ insider $\mathcal{E}$ to represent Q.
(e.g. choose $Q = \mathcal{F}^{\perp_H}$)

Write $\mathcal{E} = \mathcal{F} \oplus Q$ C^∞ direct sum

Then $\bar{\partial}_{\mathcal{E}} = \begin{pmatrix} \bar{\partial}_{\mathcal{F}} & \beta \\ 0 & \bar{\partial}_Q \end{pmatrix}$,

where $\beta \in A^{0,1}(X, \mathrm{Hom}(Q, \mathcal{F}))$, called the secondamental form.

- If β is in $\bar{\partial}(\underline{A}^0(X, \underline{Hom}(Q, \mathcal{F})))$,
  then by a gauge transformation,
  $$\mathcal{E} = \mathcal{F} \oplus Q' \quad \text{holo}^m \text{ splitting.}$$

- Isomorphism classes of $\mathcal{E}$ of the form $0 \hookrightarrow \mathcal{F} \to \mathcal{E} \to Q \to 0$ $(*)$
  is in bijection with
  $$\mathbb{P}\left(H^{0,1}_{\bar{\partial}}(X, Hom(Q, \mathcal{F}))\right)$$

- Call the extension sequence $(*)$ split
  if $[\beta] = 0$.
  iff $\exists$ an injection $Q \hookrightarrow \mathcal{E}$
  lifting the proj $\mathcal{E} \to Q$.

- For such splitting $\mathcal{E} = \mathcal{F} \oplus Q$,
  $$\bar{\partial}_{\mathcal{E}} = \begin{pmatrix} \bar{\partial}_{\mathcal{F}} & \beta \\ 0 & \bar{\partial}_Q \end{pmatrix}$$

If $H=\begin{pmatrix} H_1 & \\ & H_2 \end{pmatrix}$ w.r.t $\mathcal{E}=F\oplus Q$,

then the Chern connection ∇^H determined by $\bar\partial_E$, H is

$$\nabla^H=\begin{pmatrix} \nabla^F & \beta \\ -\beta^{*h} & \nabla^Q \end{pmatrix},$$

where ∇^F, ∇^Q are the Chern connection

β^{*h} is the adjoint of β.

(0,1)-form

(1,0)-form

The curvature of ∇^H is

$$F(\nabla^H)=\begin{pmatrix} F(\nabla^F)-\beta\wedge\beta^{*h} & \partial\beta \\ -\bar\partial\beta^{*h} & F(\nabla^Q)-\beta^{*h}\wedge\beta \end{pmatrix}.$$

§2. Moduli space of holo^{m} vector bundles

Key: To introduce stability on holo^{m} v.b.

Two motivations for stability:

① Original motivation due to Mumford is to provide the set of gauge equivalence classes of holo^{m} v.b with a "good" topology.

Mainly, the unstable ones cause "non-Hausdorff" problem.

② Turns out stability is an iff condition for a holo^{m} v.b admitting a soln to the Hermitian–Einstein eqn.

Let $\mu(E)$ denote the slope of E,

i.e. $\deg(E)/\text{rank}(E)$.

<u>Defn</u> (Mumford)

- A holo^m v.b E is called stable
semistable
if $\mu(F) < \mu(E)$ for any proper
$\leq$
holo^m subbundle F of E.

- A holo^m v.b E is called polystable
if it is a direct sum of stable
holo^m subbundles of <u>the same slope</u>.

<u>Rmk</u>: (1) Stability is preserved under gauge transformations.

(2) Stability is an open condition.

For an unstable vector bundle,

<u>Prop.</u> Given an arbitrary holo^m v.b,
$\exists!$ a Harder-Narasimhan filtration of $\mathcal{E}$,

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$$

s.t • $\mathcal{E}_i / \mathcal{E}_{i-1}$ are semistable.

• $\mu(\mathcal{E}_i / \mathcal{E}_{i-1})$ is strictly decreasing.

(Idea: Take $\mathcal{E}_1$ to be the maximal destabilizing subbundle of $\mathcal{E}$.)

For a semistable v.b,

<u>Prop.</u> Given a semistable v.b,
$\exists$ a Jordan - Hölder filtration of $\mathcal{E}$,

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$$

s.t the quotients $\mathcal{E}_i / \mathcal{E}_{i-1}$ are stable

Obviously, $\mu(\mathcal{E}_i / \mathcal{E}_{i-1})$ are the same.

Denote by $Gr(E) = \bigoplus_{i=1}^{l} \mathcal{E}_i/\mathcal{E}_{i-1}$,
(graded v.b of E)

it is polystable.

Defn. Two semistable v.b are S-equiv if their graded v.b's are gauge equiv.

Rmk: $\{\text{stable}\} \subset \{\text{polystable}\} \subset \{\text{semistable}\}$.

Denote. $\mathcal{M}^s(r,d) := \left\{ \begin{array}{l} \text{stable holo}^m \text{ v.b of} \\ \text{rk } r, \text{ deg } d \end{array} \right\} \Big/ \mathcal{G}$

$\mathcal{M}(r,d) := \left\{ \begin{array}{l} \text{polystable holo}^m \text{ v.b of} \\ \text{rk } r, \text{ deg } d \end{array} \right\} \Big/ \mathcal{G}$

$\cong \left\{ \begin{array}{l} \text{semistable holo}^m \text{ v.b of} \\ \text{rk } r, \text{ deg } d \end{array} \right\} \Big/ \text{S-equiv}$

Note that when $(r,d)=1$, $\mathcal{M}^s(r,d) = \mathcal{M}(r,d)$.

And $M^s(r,d)$ is a smooth cpt complex mfld.

Ex 1. On $\mathbb{P}^1$, by Grothendieck's thm, any holo^m v.b is of the form

$$E = \bigoplus_{i=1}^{k} \mathcal{O}(n_i).$$

Then $\mu(E) = \sum_{i=1}^{k} n_i \Big/ k$.

E is unstable unless all n_i's are equal.
polystable if all n_i's are equal.
stable only if $k=1$.

Ex 2. Consider the extension sequence

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(p) \to 0.$$

The isomorphism classes of $\mathcal{E}$ are parametrized by $\mathbb{P}(H^{0,1}_{\bar\partial}(X, \mathcal{O}(-p)))$

$$= \mathbb{P}\left(H^0(X, K(p))^*\right),$$

has $\dim_{\mathbb{C}} = g$.

<u>Claim</u>: Any non-split extension of this type is stable.

Pf: If $\mathcal{L} \hookrightarrow \mathcal{E}$ is a destabilizing line subbundle,

$$\left(\deg \mathcal{L} > \frac{\deg(\mathcal{E})}{rk(\mathcal{E})} = \frac{1}{2} \right)$$

then $\deg \mathcal{L} \geq 1$.

Can compose $\mathcal{L} \hookrightarrow \mathcal{E} \to \mathcal{O}(p)$, obtain a holo^m map $\mathcal{L} \to \mathcal{O}(p)$, which is either 0 or an isomorphism.

(i) If $\mathcal{L} \to \mathcal{O}(p)$ is 0, then $\mathcal{L} = \ker(\mathcal{E} \to \mathcal{O}(p))$

$\subset \operatorname{Ker}(\mathcal{E} \to \mathcal{O}(p))$

$\qquad\qquad\qquad\qquad \overset{\parallel}{\mathcal{O}}.$

(ii) If $L \to \mathcal{O}(p)$ is an isom,

then $\mathcal{O}(p) \to L \hookrightarrow \mathcal{E}$ is a nontrivial map

lifting the projection $\mathcal{E} \to \mathcal{O}(p)$.

So the extension splits. $\square$

Note that $\mathcal{E} = \mathcal{O} \oplus \underline{\mathcal{O}(p)}$ is unstable.

<u>Ex 3</u>: <u>Claim</u>: $0 \to \mathcal{O}_1 \to \mathcal{E} \to \mathcal{O}_2 \to 0$

with non-split extension

is strictly semistable.

Pf: • Not stable, since $\mathcal{O}_1$ has the same slope 0 as $\mathcal{E}$.

• semistable:

For any $\mathcal{L} \subset \mathcal{E}$ a holo^m line subbundle,

The induced map $\mathcal{L} \to \mathcal{O}_2$ is either zero or nontrivial.

(i) If $\mathcal{L} \to \mathcal{O}_2$ is 0, then $\mathcal{L} \hookrightarrow \mathcal{O}_1$.

$\Rightarrow \deg \mathcal{L} \leq 0$.

(ii) If $\mathcal{L} \to \mathcal{O}_2$ is nontrivial, $\Rightarrow \deg \mathcal{L} \leq 0$. ∎

Note that $Gr(\mathcal{E}) = \mathcal{O}_1 \oplus \mathcal{O}_2$.

<u>Claim</u>: $\mathcal{O}_1 \oplus \mathcal{O}_2$ is contained in the closure of the gauge orbit of $\mathcal{E}$ with non-split extension.

holo^m direct sum

Pf: Take the 1-parameter subgrp

$$g_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad (t>0).$$

Then $g_t^* \bar{\partial}_{\mathcal{E}} = g_t^* \left(\bar{\partial} + \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \right)$

$$= \bar\partial + \underset{\begin{pmatrix} t^{-1} & \\ & t\end{pmatrix}}{\underset{\shortparallel}{g_t^{-1}}} \begin{pmatrix}0&\beta\\0&0\end{pmatrix} \underset{\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}.}{\underset{\shortparallel}{g_t}} + \underbrace{g_t^{-1}\bar\partial g_t}_{0}.$$

$$= \bar\partial + \begin{pmatrix}0 & t^{-2}\beta\\ 0 & 0\end{pmatrix}$$

$$= \bar\partial \quad \text{as } t \to \infty. \qquad \square$$

Let ω_X be a Kähler form on X to be normalized s.t $\int_X \omega_X = 1$.

Thm. (Narasimhan-Seshadri)
A holo^m v.b E carries a Herm metric h satisfying the Hermitian-Einstein equ

$$\underset{\mathcal{A}^2(\mathrm{End}(E)).}{\underset{\cap}{F(\nabla h)}} = -2\pi i \cdot \mu(E) \cdot \mathrm{id}_E \cdot \omega_X$$

iff $\mathcal{E}$ is polystable.

Moreover, the soln h is unique up to multiplication by a positive constant if E is stable.

<u>Rmk</u>: From Chern-Weil theory,

$$\frac{i}{2\pi}\int_X \mathrm{Tr}(F(\nabla)) = \deg(E)$$

for any connection on E.

Use the eqn,

$$\frac{i}{2\pi}\int_X \mathrm{Tr}(F(\nabla^h)) = \frac{i}{2\pi}\int_X -2\pi i\cdot\mu(E)\cdot \mathrm{Tr}(\mathrm{id}_E)\cdot\omega_X$$

$$= \mu(E)\cdot \mathrm{rk}(E)\cdot\int_X \omega_X$$

$$= \deg(E).$$

Rmk: The first proof to N-S was algebraic and relates stable v.b with unitary reps of $\pi = \pi_1(X)$.
Donaldson presented an analytic proof.
N-S thm holds for cpt Kähler mflds of arbitrary dim, which is the Donaldson-Uhlenbeck-Yau thm.

---

§3. $\boxed{\text{Moduli space of Higgs bundles}}$

Repeat all the defns to Higgs bundles.

Defn. A Higgs bundle (E, Φ) is stable
semistable
if $\mu(F) < \mu(E)$ for any proper ϕ-inv holo^m subbundle F of E.
$\le$

polystable if $(E,\Phi) = \bigoplus_i (E_i, \Phi_i)$
stable w/ the same slope.

- H-N filtration for Higgs bundles
- J-H filtration for semistable Higgs bundles

graded polystable Higgs bundles

S-equiv

$$\mathcal{M}^{Higgs,s}(r,d) = \left\{ \begin{matrix} \text{stable Higgs bundles} \\ \text{of rk } r, \deg d \end{matrix} \right\} \Big/ G$$

$$\mathcal{M}^{Higgs}(r,d) = \left\{ \begin{matrix} \text{polystable Higgs bundles} \\ \text{of rk } r, \deg d \end{matrix} \right\} \Big/ G$$

$$\cong \left\{ \begin{matrix} \text{semistable Higgs bundles} \\ \text{of rk } r, \deg d \end{matrix} \right\} \Big/ \text{S-equiv}$$

- $\{\text{stable}\} \subset \{\text{polystable}\} \subset \{\text{semistable}\}$
- $(r,d)=1$, $\mathcal{M}^{Higgs}(r,d) = \mathcal{M}^{Higgs,s}(r,d)$.

For our intersects, we also focus on $SL(n,\mathbb{C})$ - Higgs bundles.

<u>Defn</u>. An $SL(n,\mathbb{C})$-Higgs bundle is a Higgs bundle $(\mathcal{E},\phi)$

s.t
- $\det(\mathcal{E}) \overset{\sim}{=} \mathcal{O}$
- $tr(\phi) = 0$.

Rmk: When we construct moduli space of $SL(n,\mathbb{C})$-Higgs bundles, $/\overline{\text{polystable}}$ the gauge transformation lies in $SL(n,\mathbb{C})$.

Rmk: One can even consider G-Higgs bundles for reductive Lie grps G.

e.g. $G = SO(n,\mathbb{C}), SL(n,\mathbb{R}), Sp(2n,\mathbb{R}), \ldots$ and associated stability.

Ex. Fix a holo^m line bundle $K^{\frac{1}{2}}$ of K

$\left((K^{\frac{1}{2}})^2 = K. \right)$

Let $\mathcal{E} = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ holo^m direct sum.

Claim: The Higgs bundle $(\mathcal{E}, \phi_q)$

with $\phi_q = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} : \mathcal{E} \to \mathcal{E} \otimes K$

is stable, where $q \in H^0(X, K^{\otimes 2})$.

Pf: (i) $q=0$ case. $\phi_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

The only ϕ_0-inv holo^m subbundle

is $K^{-\frac{1}{2}}$, which has deg $1-g < \underset{\deg(E)}{\overset{\shortparallel}{0}}$.

So it is stable.

(ii) Use the fact that stability is an open condition the Higgs bundle

$$\left(\mathcal{E}, \Phi_{\varepsilon q} = \begin{pmatrix} 0 & \varepsilon q \\ 1 & 0 \end{pmatrix}\right) \text{ is stable.}$$

(iii) Use $g = \begin{pmatrix} \varepsilon^{\frac{1}{4}} & \\ & \varepsilon^{-\frac{1}{4}} \end{pmatrix}$

Then (ii) means

$$\left(\mathcal{E}, g^{-1} \Phi_{\varepsilon q}\, g\right) \text{ is again stable}$$

$$\begin{gathered} \| \\ \underline{\begin{pmatrix} \varepsilon^{-\frac{1}{4}} & \\ & \varepsilon^{\frac{1}{4}} \end{pmatrix}} \begin{pmatrix} 0 & \varepsilon q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon^{\frac{1}{4}} & \\ & \varepsilon^{-\frac{1}{4}} \end{pmatrix} \\ \| \\ \varepsilon^{\frac{1}{2}} \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}. \end{gathered}$$

Claim: If (E, ϕ) is stable, so is $(E, t\phi)$ $\forall t \in \mathbb{C}^*$

Pf: Obvious. $\blacksquare$

§4. $\boxed{\text{Hitchin-Kobayashi correspondence}}$

<u>Defn</u>. Given a Higgs bundle $(\mathcal{E}, \Phi)$ over X, call a Herm metric h $\boxed{\text{harmonic}}$ if it solves the Hitchin eqn (Hitchin's self-duality eqn)

$$(*) \quad F(\nabla^h) + [\Phi \wedge \Phi^{*h}] = -2\pi i \cdot \mu(E) \cdot id_E \cdot \omega_X$$

where
- ∇^h is the Chern connection.
- $F(\nabla^h)$ is the curvature.
- Φ^{*h} is the adjoint of Φ w.r.t h,

i.e. $h(\Phi s, t) = h(s, \Phi^{*h} t)$.

Rmk: (i) Locally, if $\Phi = \varphi\, dz$ w.r.t some frame of E.

The metric presentation is h locally.

Then $\Phi^{*h} = \varphi^{*h} d\bar{z}$

$(\varphi^t h = h \overline{\varphi^{*h}} \Rightarrow \varphi^{*h} = \bar{h}^{-1} \bar{\varphi}^t \bar{h}.)$

Thus $[\Phi, \Phi^{*h}] = [\varphi, \varphi^{*h}]\, dz \wedge d\bar{z}$

Or globally, $[\Phi, \Phi^{*h}] = \phi \wedge \phi^{*h} + \phi^{*h} \wedge \phi$.

(ii) When $\deg(E) = 0$ $(\mu(E) = 0)$,

<u>Claim:</u> the Hitchin eqn $\Longrightarrow$

$D = \nabla^h + \phi + \phi^{*h}$ is flat.

Then we obtain a map from

$\{$Higgs bundles which admits harmonic metric$\}/G$

$\longrightarrow \{$flat connections$\}/G$.

Pf of Claim: $F(D) = \underbrace{F(\nabla^h) + [\phi, \phi^{*h}]} + \underbrace{\nabla^h(\phi + \phi^{*h})} = 0$

because $\nabla^h(\overset{(1,0)}{\phi} + \overset{(0,1)}{\phi^{*h}})$

$$= \underbrace{(\nabla^h)^{0,1}}\phi + \underbrace{(\nabla^h)^{1,0}\phi^{*h}}$$

$$= \bar\partial_E \phi + \underbrace{(\bar\partial_E \phi)^{*h}}$$

$$= 0 + 0 \quad \blacksquare$$

Lecture 4

Thm (Hitchin, Simpson)

A Higgs bundle $(\mathcal{E}, \phi)$ admits a harmonic metric iff $(\mathcal{E}, \phi)$ is polystable.

Moreover, it is unique up to a constant scalar if $(\mathcal{E}, \phi)$ is stable.

Rmk: • It also holds for compact Kähler mflds.

• If we strict to $(\mathcal{E}, \phi = 0)$,

this reduces to the Donaldson-Uhlenbeck-Yau thm $(\dim_{\mathbb{C}} > 1)$,

Narasimhan-Seshadri thm $(\dim_{\mathbb{C}} = 1)$.

• This thm is often referred as Hitchin-Kobayashi correspondence.

• "$\Leftarrow$" hard.

"$\Rightarrow$" easy.

Pf of "$\Rightarrow$": For a holo^m $\underline{\phi\text{-inv}}$ subbundle F of $\mathcal{E}$,

our goal is to show $\mu(F) \leq \mu(\mathcal{E})$ and rigidity.

Obtain $0 \to F \to \mathcal{E} \to Q \to 0 \quad Q = F^{\perp_H}$

So w.r.t $\mathcal{E} = F \oplus Q$ (C^∞ splitting), (H is the harmonic metric)

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_F & \beta \\ 0 & \bar{\partial}_Q \end{pmatrix}, \quad \phi = \left(\begin{array}{c|c} \Phi_1 & \gamma \\ \hline 0 & \Phi_2 \end{array}\right)$$

$$H = \begin{pmatrix} H_1 & \\ & H_2 \end{pmatrix}$$

$$[\phi, \phi^{*H}] = \phi \wedge \phi^{*H} + \phi^{*H} \wedge \phi$$

So we have

$$\nabla^H = \begin{pmatrix} \nabla_F & \beta \\ -\beta^{*h} & \nabla_Q \end{pmatrix} \qquad \phi^{*h} = \left(\begin{array}{c|c} \Phi_1^{*H} & 0 \\ \hline \gamma^{*H} & \Phi_2^{*H} \end{array}\right)$$

$$[\phi, \phi^{*H}]_{(1,1)} = \phi_1 \wedge \phi_1^{*H} + \gamma \wedge \gamma^{*H} + \phi_1^{*H} \wedge \phi_1$$

Then rewrite the Hitchin eqn

$$F(\nabla^h) + [\phi, \phi^{*h}] = -2\pi i \cdot \mu_E \cdot Id_E \cdot \omega_X$$

w.r.t $\mathcal{E} = F \oplus Q$,

$$\Rightarrow \left(\begin{array}{c|c} F(\nabla^F) - \beta \wedge \beta^{*h} + [\phi_1, \phi_1^{*H}] + \gamma \wedge \gamma^{*H} & * \\ \hline * & * \end{array}\right) = -2\pi i \cdot \mu_E \cdot Id_E \cdot \omega_X.$$

$$\Rightarrow \frac{i}{2\pi} \int_X Tr(F(\nabla^F)) - Tr(\overset{(0,1)\text{-form}}{\beta} \wedge \beta^{*h}) + \underbrace{Tr[\phi_1, \phi_1^{*H}]} + \underline{Tr(\overset{(1,0)\text{-form}}{\gamma} \wedge \gamma^{*h})}$$

$$-i \cdot d\bar{z} \wedge dz \qquad\qquad i\, dz \wedge d\bar{z}.$$

$$= \cancel{-2\pi i} \cdot \mu_E \cdot \int_X Tr(Id_E|_F)\, \omega_X$$

$$\Rightarrow \deg(F) + \|\beta\|^2_{L^2} + 0 + \|\gamma\|^2_{L^2} = \mu_E \cdot rk(E) \cdot rank(F)$$

$$\Rightarrow \quad \mu(F) \leq \mu(E).$$

Moreover, the equality holds iff $\beta=0$, $\gamma=0$.

iff $(E,\phi)=(F,\phi_1)\oplus(Q,\Phi_2)$.

$\Rightarrow$ (E,ϕ) is polystable. $\square$

§5. $\boxed{\text{Hitchin moduli space}}$

Sometimes it's natural to fix a background Herm metric h_0 on E and consider the following system of p.d.e.

$$(**)\begin{cases} F(A)+[\phi,\phi^{*h_0}] = -2\pi i\cdot\mu_E\cdot \mathrm{id}_E\cdot\omega_X \\ \bar{\partial}_A\phi = 0. \qquad (\bar{\partial}_A = A^{0,1} = d_A'') \end{cases}$$

about (A,ϕ), A is a unitary connection w.r.t h_0.

$\phi\in \mathcal{A}^{1,0}(X,\mathrm{End}(E))$.

These are also called Hitchin's self-duality eqn.

<u>Claim</u>: $(*)\Leftrightarrow(**)$.

Assume we are giving with $(\bar{\partial}_E, \phi)$ and a harmonic h solving $(*)$.

We find a gauge transformation g s.t
$$h_0 = g^* h.$$

Then the gauge transformed pair $g^*(\nabla^h, \phi)$ is a soln of $(**)$.

The other way is $[(A, \phi)] \mapsto [(\bar{\partial}_A, \phi)]$. ▨

$\boxed{\text{Hitchin moduli space}}$

$$\mathcal{M}^{\text{self-dual}}(r, d) := \left\{ (A, \Phi) \mid \text{irreducible soln of } (**) \right\} \Big/ \mathcal{G}(E, h_0)$$

Here, irreducible means solns does not split into solns on any nontrivial decomp of (E, h_0) as a direct sum of Hermitian v.b.

- One can show that $\mathcal{M}^{\text{self-dual}}(r,d)$ is a finite-dim smooth mfld.

- $\mathcal{M}^{Higgs, s}(r,d) \underset{\simeq}{\xrightarrow{homeo}} \mathcal{M}^{\text{self-dual}}(r,d)$.

---

§6. $\mathbb{C}^*$- action on $\mathcal{M}^{Higgs}$ and S^1- action on $\mathcal{M}^{\text{self-dual}}$.

$\mathbb{C}^*$-action:
$$\mathbb{C}^* \times \mathcal{M}^{Higgs} \to \mathcal{M}^{Higgs}$$
$$(t, [(E,\Phi)]) \mapsto [(E, t\Phi)]$$

S^1-action:
$$S^1 \times \mathcal{M}^{\text{self-dual}} \to \mathcal{M}^{\text{self-dual}}$$
$$(e^{i\theta}, [(A,\Phi)]) \mapsto [(A, e^{i\theta}\Phi)]$$

Rmk: Under the correspondence between $\mathcal{M}^{Higgs}$ with $\mathcal{M}^{\text{self-dual}}$, the S^1- action coincides.

Now we want to characterize the fixed pts of $\mathbb{C}^*$-actions / S^1- actions in $\mathcal{M}^{Higgs}$.

Defn. A Hodge bundle is a Higgs bundle of the form
$$\mathcal{E} = \mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_k$$
$$\theta: \mathcal{E} \to \mathcal{E} \otimes K$$
$$\mathcal{E}_j \to \mathcal{E}_{j+1} \otimes K \quad \forall\, 0 \le j \le k-1.$$

$$\mathcal{E}_k \to 0.$$

- $\mathcal{E}_0 \xrightarrow{\theta_0} \mathcal{E}_1 \xrightarrow{\theta_1} \mathcal{E}_2 \xrightarrow{\theta_2} \cdots \xrightarrow{\theta_{k-1}} \mathcal{E}_k.$

Prop. (Hitchin / Simpson) $(\mathcal{E},\theta)$ is a Hodge bundle
iff $(\mathcal{E},\theta)$ is fixed by the $\mathbb{C}^*$-action
iff $(\mathcal{E},\theta)$ is fixed by the S^1-action.

Pf: Assume $(\mathcal{E},\theta)$ is fixed by some $t \in S^1$ which is not a root of unity.
goal is to show it is a Hodge bundle.
By the defn of being fixed under some t,
$\exists g \in \mathcal{G}(E)$ s.t $\underbrace{g\bar{\partial}_E g^{-1} = \bar{\partial}_E}$, $g\theta g^{-1} = t\theta$

$\Downarrow$

g is holo^m w.r.t $\bar{\partial}_E$.

$\Downarrow$

coefficients of $\det(\lambda I - g)$ are holo^m fcns on X, thus constant.

$\Downarrow$

$\det(\lambda I - g)$ is polynomial of λ with constant coefficients.

$\Downarrow$

g's eigenvalues are constant.

So we have a decomposition

$$E = \bigoplus E_\lambda,$$

where $E_\lambda = \operatorname{Ker}(g - \lambda \, Id_E)^{rk E}$

is the generalized eigenspace correspond to λ.

From $g\theta g^{-1} = t\theta$,

$$\Rightarrow \quad g\theta = t\theta g$$

$$\Rightarrow \quad g\theta - \lambda t \cdot \theta = t\theta g - \lambda t \cdot \theta$$

$$\Rightarrow \quad (g - \lambda t \cdot Id_E) \cdot \theta = t\theta (g - \lambda Id_E)$$

Repeat

$$\Rightarrow \quad (g - \lambda t \cdot Id_E)^{rkE} \cdot \theta = t^{rkE} \theta (g - \lambda Id_E)^{rkE}$$

$\Rightarrow$ θ maps E_λ to $E_{t\lambda} \otimes K$.

Since t is not a root of unity, and 0 is not an eigenvalue,

there are eigenvalues $\lambda_1, \cdots, \lambda_s$ and integers $k_1, \cdots, k_s$ so that

$\forall\, 1 \le j \le s$, $\lambda_j, t\lambda_j, \cdots, t^{k_j}\lambda_j$ are eigenvalues of g

but $t^{-1}\lambda_j$, $t^{k_j+1}\lambda_j$ are not.

Then for each $1 \le j \le s$,

$\bigoplus_{i=0}^{k_j} E_{t^i\lambda_j}$ is a Hodge bundle with the induced action of θ.

Combine them together, we give $(\mathcal{E},\theta)$ a system of Hodge bundle str.

Conversely, suppose $(\mathcal{E},\theta)$ is a Hodge bundle, then it's $\mathbb{C}^*$-inv.

$\forall\, t\in\mathbb{C}^*$, take $g_t = \begin{pmatrix} t^0\,\mathrm{id}_{E_0} & & & \\ & t^1\cdot\mathrm{id}_{E_1} & & \\ & & \ddots & \\ & & & t^k\cdot\mathrm{id}_{E_k} \end{pmatrix}$

$$\Rightarrow g_t\,\theta\, g_t^{-1} = \begin{pmatrix} t^0 & & & \\ & t^1 & & \\ & & \ddots & \\ & & & t^k \end{pmatrix}\begin{pmatrix} 0 & & & & \\ \theta_0 & 0 & & & \\ & \theta_1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \theta_{k-1} & 0 \end{pmatrix}\begin{pmatrix} t^0 & & & \\ & t^{-1} & & \\ & & \ddots & \\ & & & t^{-k} \end{pmatrix}$$

$$= t\cdot\begin{pmatrix} 0 & & & \\ \theta_0 & 0 & & \\ & \ddots & \ddots & \\ & & \theta_{k-1} & 0 \end{pmatrix} = t\theta.$$

$\square$

Hitchin fibration, $\mathcal{M}^{\text{self-dual}}(r,d) \ni [(A,\Phi)]$

$$\mathcal{H}: \mathcal{M}^{\text{Higgs},s}_{(r,d)} \longrightarrow \mathcal{B} = \underset{g}{H^0(X,K)} \oplus \underset{3(g-1)}{H^0(X,K^{\otimes 2})} \oplus \underset{5(g-1)}{\cdots} \oplus \underset{(2r-1)(g-1)}{H^0(X,K^{\otimes r})}$$

$$[(E,\phi)] \longmapsto \left(\operatorname{tr}(\phi), \operatorname{tr}(\phi^2), \cdots, \operatorname{tr}(\phi^r)\right)$$

Rmk: • One can also replace trace with Ad-inv symmetric polynomials on $\mathfrak{gl}(r,\mathbb{C})$.

e.g. the coeffients of $\det(\lambda I - \phi)$.

• $\mathcal{H}((E,\phi))$ determines the eigen 1-forms of ϕ, "spectral" of ϕ.

• $\mathcal{B}$ is called Hitchin base.

$\dim_{\mathbb{C}} \mathcal{B} = g + (r^2-1)(g-1)$.

Lemma (Simpson) (Bounded spectral implies bounded norm of Higgs field)

Fix a background Kähler metric g on X.

Given C_1, $\exists$ a constant C_2 s.t

if (E,ϕ) is a polystable Higgs bundle whose eigenvalues of ϕ have norm (w.r.t g) $\leq C_1$,

then for a harmonic metric h on (E,ϕ),

$$|\phi|_{h,g} \leq C_2.$$

$$\boxed{\Delta \log|\phi|^2_{g,h} \geq \frac{|[\phi,\phi^*]|^2}{|\phi|^2} \geq -C|\phi|^2_{g,h} + C'}$$

Prop. (Hitchin) The Hitchin fibration $\mathcal{H}$ is proper.

"Pf" Prove from the Hitchin moduli viewpt.

Since $F(A) + [\phi, \phi^{*_{h_0}}] = 0$

$$\Rightarrow F(A) = -\phi \wedge \phi^{*_{h_0}} - \phi^{*_{h_0}} \wedge \phi$$

By Lemma. $\Rightarrow$ $F(A)$ is bounded. (i.e. the eigenvalue of ϕ_j are bounded)

For a sequence of (A_j, ϕ_j) s.t $H((A_j, \phi_j)) \le C$

By Uhlenbeck's weak compactness thm,

(If $F(A_j)$ has uniform L^p bound,)

$\exists$ a sequence of unitary gauge transf $g_j \in L^p_2$,

and a smooth connection A_∞, s.t

(after passing to subsequence),

$$g_j(A_j) \to A_\infty$$

weakly in L^p_1 and strongly in L^p.

By the boundedness of ϕ,

$\exists g_j$ s.t $g_j(\phi_j) \to \phi_\infty$.

weakly in L^p_1 and strongly in L^p.

$$\Rightarrow g_j(A_j, \phi_j) \to (A_\infty, \phi_\infty).$$

The rest relies on (A_j, ϕ_j) are solns to (**)

standard elliptic theory,

Sobolev embedding, p can be large enough.

$\Longrightarrow g_j(A_j, \phi_j) \to (A_\infty, \phi_\infty)$ C^α. $\blacksquare$

smooth.