

Lecture 5:

Explain the hyperKähler geometry on
 $M^{\text{self-dual}, s}(r, d)$

§1. Quick tour on symplectic geometry

Defn. A symplectic mfld is a pair (X, ω) ,

- X is a mfld

- $\omega \in \wedge^2(X, \mathbb{R})$ is non-deg, i.e

$$\forall x \in X, \quad \omega_x: T_x X \rightarrow T_x X^*$$

$$v \mapsto i_v \omega := \omega(v, \cdot)$$

is an isom.

- $d\omega = 0$.

Rmk: When we deal with infinite dim mfld,
only ask $\omega_x: T_x X \rightarrow T_x X^*$ is injective
for symplectic str.

Defn. (Moment map).

Suppose (X, ω) is a symplectic mfld.

Let G be a real Lie grp acting on (X, ω) .

Let $\mathfrak{g} = \text{Lie}(G)$ and let $P: \mathfrak{g} \rightarrow \text{Vect}(X)$

$$V \mapsto \left. \frac{d}{dt} \right|_{t=0} \exp(tV) \cdot P.$$

be the infinitesimal action.

Suppose given a fcn $\mu: X \rightarrow \mathfrak{g}^*$,

write $\mu_Z = \mu(Z) : X \rightarrow \mathbb{R}$, $\forall Z \in \mathfrak{g}$

We say μ is a moment map for the G -action

- $\bullet \forall Z \in \mathfrak{g}, i_{p(Z)} \omega = d\mu_Z.$
- $\bullet \mu$ is G -equivariant.

Rmk: • Moment maps do not always exist.

A necessary condition is that G -action preserves ω , i.e. $L_{p(Z)} \omega = 0, \forall Z \in \mathfrak{g}.$

(This follows from Cartan formula:

$$L_{p(Z)} \omega = \underline{d(i_{p(Z)} \omega)} + \overbrace{i_{p(Z)}(d\omega)}^0$$

Still not a sufficient condition.

• Call G -action having moment maps hamiltonian.

Defn / Prop (Symplectic quotient is symplectic)

Given $(X, \omega, G, \mu).$

Then the symplectic quotient is

$$X//G := \underline{\mu^{-1}(0)} / G.$$

$\mu^{-1}(a)/G$
also works
if a is
a regular
value.

There is a symplectic form $\omega_{X//G}$ on $X//G$ with the property $\pi^* \omega_{X//G} = l^* \omega$ by ($\because \mu^{-1}(0) \hookrightarrow X$).

This quotient is called the Marsden-Weinstein quotient.

Ex 1. $G \curvearrowright N$ freely.

$\rightsquigarrow G \curvearrowright T^*N$.

On T^*N , there is a natural symplectic form ω :

$\omega = d\lambda$, λ is the Liouville 1-form on T^*N

given by $\lambda_{(x,p)}(V) := p(\underbrace{\pi_*(V)}_{T_x N})$.

The moment map is given by

$$\mu_Z := \lambda(p_Z). \quad \forall Z \in \mathfrak{g}$$

• μ_Z is G -equivariant.

In fact, $T^*N // G := \mu^{-1}(0)/G \cong T^*(N/G)$.

Ex 2. $X = \mathbb{C}^n$, $\omega = \frac{i}{2} \sum_i dz_i \wedge d\bar{z}_i$

Consider a $U(1)$ -action: $z_i \rightarrow e^{i\theta} z_i$, $i=1, \dots, n$.

A moment map $\mu = -\frac{1}{2} \sum_i |z_i|^2 + c$, $c > 0$.

Then $X // U(1) := \mu^{-1}(0)/U(1) = \mathbb{C}\mathbb{P}^{n-1}$

§2.

Hyperkähler structure on the affine space

$$\mathcal{E} = \{(A, \phi)\}$$

Let \mathcal{E} be the infinite-dim space

$\mathcal{E} = \{(A, \phi) \mid A \text{ unitary connection on } (E, h_0),$
 $\phi \in \mathcal{A}^{1,0}(X, \text{End}(E))\}$

For $(A, \Phi) \in \mathcal{L}$, $T_{(A, \Phi)} \mathcal{L} \cong \mathcal{A}^{0,1}(X, \text{End}(E)) \oplus \mathcal{A}^{1,0}(X, \text{End}(E))$

- A natural complex structure on \mathcal{L} is given by

$$I(\dot{\alpha}, \dot{\phi}) = (i\dot{\alpha}, i\dot{\phi})$$

- There is a Riemannian metric on \mathcal{L} given by

$$g((\dot{\alpha}_1, \dot{\phi}_1), (\dot{\alpha}_2, \dot{\phi}_2)) = \text{Re } i \int_X \text{tr}(\dot{\phi}_1 \wedge \dot{\phi}_2^* - \dot{\alpha}_1 \wedge \dot{\alpha}_2^*)$$

$$(= \int_X \langle \dot{\phi}_1, \dot{\phi}_2 \rangle + \langle \dot{\alpha}_1, \dot{\alpha}_2 \rangle d\text{vol}_X)$$

If turns out g is Kähler w.r.t I , $w_I^{(., .)} = g(I \cdot, \cdot)$

Defn. Let (X, I) be a complex mfld.

A Hermitian metric on (X, I) is a Riem metric g s.t $g(V, W) = g(IV, IW)$.

A Herm metric g on (X, I) is Kähler if $\nabla^g I = 0$.

For Kähler metric g , define a 2-form $u \in \mathcal{A}^{1,1}(X)$,

by $u(V, W) = g(IV, IW)$

If is a symplectic 2-form.

- There is also a complex symplectic form u^c on \mathcal{L}

given by

$$\Omega((\dot{\alpha}_1, \dot{\phi}_1), (\dot{\alpha}_2, \dot{\phi}_2)) = \int_X \text{tr}(\dot{\phi}_2 \wedge \dot{\alpha}_1 - \dot{\phi}_1 \wedge \dot{\alpha}_2)$$

(Here, complex symplectic means Ω is a $(2,0)$ -form, nondeg, \mathbb{R} holm w.r.t I .)

$$\begin{aligned}\Omega(Iv, w) &= i\Omega(v, w) \\ \Omega(v, Iw) &= i\Omega(v, w).\end{aligned}$$

- Rmk:
- nondegeneracy follows from trace is a nondeg pairing on $gl(n, \mathbb{C})$.
 - hol^M follows from constant coefficients.
So is the Kähler property of g .
 - One can also view \mathcal{E} as the cotangent bundle of $\mathcal{A} = \{\text{space of hol}^M \text{ structures on } E\}$
by $\langle \dot{\alpha}, \dot{\phi} \rangle = \int_X (\dot{\alpha} \wedge \dot{\phi})$.
 - $\Omega = d\lambda$.
 - $\Omega = w_J + iw_K$ by taking real and imaginary parts.
Using g again, we obtain
two more complex structures J, K
by $w_J(\cdot, \cdot) = g(J\cdot, \cdot)$, $w_K(\cdot, \cdot) = g(K\cdot, \cdot)$.
Again, J, K are integrable.
 - What are J, K ?
 $J(A, B) = (iB^*, -iA^*)$
 $K(A, B) = (-B^*, A^*)$.

The calculations for J is as follows:

J is defined st $\omega_J(\cdot, \cdot) = g(J\cdot, \cdot)$

Suppose $J(A, B) = (E, F)$, then

$$g(J(A, B), (\dot{\alpha}, \dot{\phi})) = \omega_J((A, B), (\dot{\alpha}, \dot{\phi}))$$

$$g((E, F), (\dot{\alpha}, \dot{\phi}))$$

$$\text{Re } i \int_X \text{tr}(F \dot{\alpha}^* - E \wedge \dot{\alpha}^*)$$

$$\text{Re } \int_X \text{tr}(\dot{\phi} \wedge A - B \wedge \dot{\alpha})$$

Use $\text{Re } \text{tr}(\alpha \wedge \beta) \underset{\text{if } \alpha, \beta \text{ are 1-forms of different type.}}{\parallel} \text{Re } \text{tr}(\beta^* \wedge \alpha^*)$

$$\text{Re } \int_X \text{tr}(\dot{\phi} \wedge (iF)^* - \dot{\alpha} \wedge (iE)^*)$$

$$\text{Re } \int_X \text{tr}(\dot{\phi} \wedge A + \dot{\alpha} \wedge B)$$

Compare both sides, we obtain

$$\begin{cases} (iF)^* = A \\ -(iE)^* = B \end{cases} \Rightarrow \begin{cases} F = -iA^* \\ E = iB^* \end{cases}$$

$$\text{So } J(A, B) = (E, F) = (iB^*, -iA^*)$$

$$\text{And } J^2(A, B) = J(iB^*, -iA^*) = -(A, B)$$

$$\text{So } J^2 = -\text{Id}.$$

$$\text{Since } \omega_K(\cdot, \cdot) = -\omega_J(i\cdot, \cdot) = -\omega_J(\overset{\parallel}{I}\cdot, \cdot),$$

$$g(K\cdot, \cdot) \qquad \qquad \qquad -g(JI\cdot, \cdot)$$

$$\text{So } K = -JI = IJ \text{ and } K(A, B) = (-B^*, A^*). \quad \square$$

Defn. A hyperKähler mfld is (X, g, I, J, K) w/ $IJ=K$
 s.t. (X, g, \cdot) is Kähler. $\cdot \in \{I, J, K\}$

Rmk: • w_I, w_J, w_K .

- In fact, then $aI + bJ + cK, \forall (a, b, c) \in S^2$
 is also a complex strn X .

Now we can state:

Prop. (E, g, I, J, K) is a hyperKähler mfld
 of infinite dim.

§3. Interpret the Hitchin eqn (**) as three moment maps

Prop. The action of unitary gauge grp G^U on E
 is hamiltonian for all the Kähler forms w_I, w_J, w_K .
 And the corresponding moment map for w_I, w_J, w_K
 are resp

$$m_I(A, \Phi) = F(A) + [\Phi, \Phi^*]^J$$

$$m^c(A, \Phi) = (\underline{w_J + i w_K})(A, \Phi) = \bar{\partial}_A \Phi.$$

decomposition w.r.t $g^U(r, \mathbb{C}) = U(r) \oplus iV(r)$

That is, (A, Φ) solving the Hitchin eqn $\text{Herm}(r)$
 $\Leftrightarrow (A, \Phi) \in M_I^{-1}(-2\pi i \cdot \mu_E \cdot \text{id}_E \cdot w_X) \cap M_J^{-1}(0) \cap M_K^{-1}(0)$.

Assuming this proposition, we obtain the following
then after some extra work.

Thm (Hitchin)

The induced metric g on

$$\mathcal{M}_{(r,d)}^{\text{self-dual}, s} = \mathcal{M}_I^{-1}(-2\pi i \cdot \text{Id}_E \cdot u_X) \cap \mathcal{M}_J^{(0)} \cap \mathcal{M}_K^{(0)} \xrightarrow{G^u}$$

is hyperKähler.

Rmk: When $(r,d)=1$, the metric g is complete.

Before the proof of Prop, we explain:

- Note that $\mu_0 \in \mathcal{A}^{1,1}(X, \mathfrak{u}(E))$

$$g^u = \text{Lie}(G^u) = \mathcal{A}^0(X, \mathfrak{u}(E))$$

We identify $(g^u)^*$ with $\mathcal{A}^{1,1}(X, \mathfrak{u}(E))$ by

$$\mathcal{A}^0(X, \mathfrak{u}(E)) \times \mathcal{A}^{1,1}(X, \mathfrak{u}(E)) \rightarrow \mathbb{C}$$

$$(\varphi, \Psi) \mapsto \int_X \text{tr}(\varphi \Psi).$$

Similarly, we can identify g^* with $\mathcal{A}^{1,1}(X, \text{End}(E))$.

$$\text{for } g = \text{Lie}(G) = \mathcal{A}^0(X, \text{End}(E))$$

$$\text{So } (\mu_\bullet)_\varphi(A, \phi) = \int_X \text{tr}(\mu_\bullet(A, \phi) \varphi)$$

for $\varphi \in \mathcal{F}^u$

$$\leadsto (\mu_\bullet) : \mathcal{C} \rightarrow \mathcal{F}^*$$

- Let $\varphi \in \mathcal{F}^u = \mathcal{D}^0(X, \mu(E))$,

it defines the vector field

$$X = (\dot{\alpha}, \dot{\phi}) = (\bar{\partial}_A \varphi, [\underline{\Phi}, \varphi]) \in T_{(A, \underline{\Phi})} \mathcal{C}$$

This follows from $\dot{\alpha} = \frac{d}{dt}|_{t=0} \exp(-t\varphi) \bar{\partial}_A \exp(t\varphi)$

$$\dot{\phi} = \frac{d}{dt}|_{t=0} \exp(-t\varphi) \underline{\Phi} \exp(t\varphi).$$

Pf of Prop on moment maps:

Part I: Show $\bar{\partial}_A \underline{\Phi}$ is the moment map μ^c for $\omega^c = \omega_J + i\omega_K$ in the sense that

$$(\mu^c)_\varphi = \int_X \text{tr}(\bar{\partial}_A \underline{\Phi} \cdot \varphi) : \mathcal{C} \rightarrow \mathbb{C}.$$

$$\text{Pf: } (\underline{i}_{P_\varphi} \omega^c)_{(A, \underline{\Phi})}(\dot{\alpha}, \dot{\phi}) = \int_X \text{tr}(\dot{\phi} \wedge \bar{\partial}_A \underline{\Phi} - \underbrace{[\underline{\Phi}, \underline{\Phi}] \wedge \dot{\alpha}}_{+ [\underline{\Phi}, \underline{\Phi}] \wedge \dot{\alpha}})$$

$$= \int_X \text{tr}(\dot{\phi} \wedge d_A \underline{\Phi} + \varphi [\underline{\Phi}, \dot{\alpha}]).$$

$$= \int_X \text{tr}(d_A \dot{\phi} \cdot \varphi + \varphi [\underline{\Phi}, \dot{\alpha}])$$

$$= \int_X \text{tr}(\bar{\partial}_A \dot{\phi} \cdot \varphi + [\dot{\alpha}, \underline{\Phi}] \varphi)$$

$$\omega^c((\dot{\alpha}_1, \dot{\phi}_1), (\dot{\alpha}_2, \dot{\phi}_2))$$

$$= \int_X \text{tr}(\dot{\phi}_2 \wedge \dot{\alpha}_1 - \dot{\phi}_1 \wedge \dot{\alpha}_2)$$

Lemma. $\text{Tr}([\bar{x}, \bar{y}]z) = \text{Tr}(x[\bar{y}, z])$

$\alpha, \beta \in \Omega^1(X, \text{End}(E))$ also holds for forms.

$$[\alpha, \beta] = \alpha \wedge \beta + \beta \wedge \alpha.$$

$$\begin{aligned} &= \int_X \text{tr}((\bar{\partial}_A \dot{\phi} + [\dot{\alpha}, \dot{\phi}]) \psi) \\ &= \int_X \text{tr}((\bar{\partial}_A \dot{\phi}) \psi) \\ &= d \left(\int_X \text{tr}(\bar{\partial}_A \dot{\phi} \cdot \psi) \right) (\dot{\phi}) \end{aligned}$$

- $\bar{\partial}_A \dot{\phi}$ is G^u -equivariant.

Part II: $\frac{1}{2}(F(A) + [\phi, \phi^*])$ is the moment map for w_I .

Pf: $(i_{p_4} w_I)_{(A, \dot{\phi})}(\dot{\alpha}, \dot{\phi}) = w_I(p_4, (\dot{\alpha}, \dot{\phi}))$

$$= g(I p_4, (\dot{\alpha}, \dot{\phi}))$$

$$= g(i \bar{\partial}_A \psi, i [\dot{\phi}, \psi]), (\dot{\alpha}, \dot{\phi})$$

$$= \text{Re } i \cdot \int_X \text{tr}(i [\dot{\phi}, \psi] \wedge (\dot{\phi})^* - i \bar{\partial}_A \psi \wedge \dot{\alpha}^*)$$

$$= \text{Re} \int_X ([\psi, \dot{\phi}] \wedge (\dot{\phi})^* + d_A \psi \wedge \dot{\alpha}^*)$$

$$= \text{Re} \int_X \psi [\dot{\phi}, (\dot{\phi})^*] - \psi \cdot d_A (\dot{\alpha}^*)$$

$$\boxed{g((\dot{\alpha}, \dot{\phi}), (\dot{\alpha}_2, \dot{\phi}_2)) = \text{Re} i \int_X \text{tr}(\dot{\phi}_1 \wedge \dot{\phi}_2^* - \dot{\alpha}_1 \wedge \dot{\alpha}_2^*)}$$

$$= \frac{1}{2} \int_X \text{tr}(\psi [\dot{\phi}, \dot{\phi}^*] + (\psi [\dot{\phi}, \dot{\phi}^*])^* - \psi \cdot d_A (\dot{\alpha}^*) - (\psi \cdot d_A \dot{\alpha}^*)^*)$$

Use $\psi^* = -\psi$

$$= \frac{1}{2} \int_X \text{tr} (\varphi \cdot ([\dot{\phi}, \dot{\phi}^*] + [\dot{\phi}, \phi^*]) - \varphi (d_A \ddot{\alpha}^* - d_A \dot{\alpha}))$$

$$= \frac{1}{2} \int_X \text{tr} \left((d_A(\dot{\alpha} - \dot{\alpha}^*) + [\dot{\phi}, \dot{\phi}^*]) \varphi \right)$$

$$= \frac{1}{2} \int_X \text{tr} \left(\underbrace{(F(A) + [\dot{\phi}, \dot{\phi}^*])}_{\text{F}(A)} \varphi \right).$$

Here we use $\frac{d}{dt}|_{t=0} F(A+t\eta) = d_A \eta$ and $\eta = \dot{\alpha} - \dot{\alpha}^*$.

$$\begin{aligned} & \frac{d}{dt}|_{t=0} \left(d(A+t\eta) + \frac{1}{2} [A+t\eta, A+t\eta] \right) \\ &= d\eta + [A, \eta] = d_A \eta. \end{aligned}$$



Lecture 6

M^{Higgs} is an algebraic completely integrable system.

Part I: Completely integrable system (holo^m version.)

First, recall $\mathcal{C} = \{(\mathbf{A}, \phi) \mid \mathbf{A} \text{ unitary connection}$
 $\text{on } (E, h_0),$
 $\phi \in A^{1,0}(X, \text{End}(E))\}$

$$\cong \overline{\{\text{holo}^m \text{ str's on } E\}} \times A^{1,0}(X, \text{End}(E)).$$

On \mathcal{C} , there is g , w_I , $w^c = w_J + i w_K$
 I, J, K .

Also constructed moment maps for
the action of G^u ,

$$w_I, \quad w_J + i w_K = w_c$$

$$F(A) + \frac{i}{2} [\phi, \phi^*] \qquad \qquad \bar{\partial}_A \phi$$

Observe that w_c is not just moment map for G^u ,
but also moment map for G .

So we can consider the symplectic quotient

$$\frac{\{\text{polystable Higgs bundles}\} \mathcal{M}_C^{-1}(0)}{G}$$

$\mathcal{M}_C^{-1}(0) = \{(\bar{\partial}E, \phi) \mid \bar{\partial}E\phi = 0\}.$

$\mathbb{G} \subset G$

$M^{\text{Higgs}} \quad ||$

So on M^{Higgs} , we have descended

$$\begin{matrix} w^C & , & I \\ \uparrow & & \uparrow \\ \text{holo}^m \text{ symplectic form} & & \text{complex str.} \end{matrix}$$

Defn. On a symplectic mfld (M, w)
 and $f: M \rightarrow \mathbb{R}$ a smooth fn,
 the vector field X_f satisfying $i_{X_f} w = df$
 is called the Hamiltonian vector field
 associated to f .

The fn f is called a Hamiltonian for
 the Hamiltonian v.f X_f .

Defn. The Poisson bracket of two fns f and g is defined by

$$\{f, g\} := X_f \cdot g = -X_g \cdot f = -\{g, f\}$$

$$= w(X_f, X_g).$$

Two fns are said to Poisson commute if $\{f, g\} = 0$.

e.g. $w = \sum dx_i \wedge dy_i$, f and g are fns of y_1, \dots, y_n alone.

$$\text{Then } df = \sum \frac{\partial f}{\partial y_i} dy_i = i(X_f) (\sum dx_i \wedge dy_i)$$

$$\Rightarrow X_f = \sum \frac{\partial f}{\partial y_i} \frac{\partial}{\partial x_i}.$$

$$\text{So } \{f, g\} = X_f \cdot g = \sum \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} = 0.$$

e.g. Given the data (M, w, G, μ) , moment map

If g, h are two G -inv Poisson-commuting fns on M ,

then g, h descend to Poisson-commuting fns \tilde{g}, \tilde{h} on $M^1(0)/G (= M/G)$.

(Exercise.)

Defn. A symplectic mfld M of dim $\underline{2n}$ is said to be a completely integrable Hamiltonian system if \exists fcns $f_1, \dots, f_n : M \rightarrow \mathbb{R}$ which Poisson-commute and for which $df_1 \wedge \dots \wedge df_n$ is generically nonzero.

We use the complex / holo^m version.

Consider the Hitchin fibration

$$H: M^{\text{Higgs}}(\text{SL}(n, \mathbb{C})) \rightarrow \mathcal{B} = H^0(X, K^2) \oplus \dots \oplus H^0(X, K^n)$$

$$[(E, \phi)] \mapsto (\text{tr} \phi^2, \text{tr} \phi^3, \dots, \text{tr} \phi^n).$$

Thm (Hitchin) Let $\alpha_1, \dots, \alpha_m = (n-1)(g-1)$ be a basis of the vector space \mathcal{B}^* .

Then the fcns $f_i = \alpha_i \circ H : M^{\text{Higgs}}(\text{SL}(n, \mathbb{C})) \rightarrow \mathbb{C}$
 $(1 \leq i \leq m.)$

Poisson commute w.r.t the Poisson bracket determined by the holo^m symplectic str ω^c on $M^{\text{Higgs}}(\text{SL}(n, \mathbb{C}))$.

(Observe that f_i 's are holom.).

(Rmk: By the fact that $\dim_{\mathbb{C}} \mathcal{M}^{\text{Higgs}}(\text{SL}(n, \mathbb{C}))$

$$= 2 \dim_{\mathbb{C}} \mathcal{B}$$

$$= (n^2 - 1)(2g - 2),$$

we conclude that $(\mathcal{M}^{\text{Higgs}}(\text{SL}(n, \mathbb{C})), w^c)$ is
a completely integrable system. (complex version.)

Pf: Only prove for the case $\alpha_s \in H^0(X, K^{i(s)})^*$.

(This means for each $(\leq s \leq m,$

$$\exists i = i(s) \text{ s.t } \alpha_s \in H^0(X, K^{i(s)})^*$$

By Serre duality, $H^0(X, \overset{\alpha_s}{K^{i(s)}})^* \xrightarrow{\cong} H^1(X, \overset{\beta_s}{K^{1-i(s)}})$

i.e. $\alpha_s(q_j) := \begin{cases} \int_X q_j \wedge \beta_s & j = i(s) \\ 0 & (\text{if } j \neq i(s)). \end{cases}$

Now think of $\tilde{f}_s = \alpha_s \circ H : \mathcal{E} \rightarrow \mathbb{C}$,

they are G -invariant.

It's enough to show \tilde{f}_s' Poisson commute.

$$\begin{aligned}
\tilde{f}_S &= \alpha_S \circ H((A, \phi)) \\
&= \int_X \text{tr}(\phi^{i(S)}) \wedge \beta_S. \\
\Rightarrow d\tilde{f}_S(\dot{A}, \dot{\Phi}) &= \int_X i(S) \cdot \text{tr}(\phi^{i(S)-1} \cdot \dot{\Phi}) \wedge \beta_S. \\
&= w^c \left(\underbrace{(i(S) \Phi^{i(S)-1} \cdot \beta_S, 0)}_{\text{---}}, (\dot{A}, \dot{\Phi}) \right). \\
&= (i(X_{\tilde{f}_S}^c) w_c)(\dot{A}, \dot{\Phi}). \\
\Rightarrow X_{\tilde{f}_S}^c &= (i(S) \Phi^{i(S)-1} \cdot \beta_S, 0) \\
\{X_{\tilde{f}_S}^c, X_{\tilde{f}_t}^c\} &= w^c(X_{\tilde{f}_S}^c, X_{\tilde{f}_t}^c) = 0.
\end{aligned}$$

□

Part II: algebraic.

"Generic fibers of $H: M^{\text{Higgs}} \rightarrow B$
are abelian variety, that is, complex torus."

Plus the fact that H is proper.

Then we obtain

$$(M^{\text{Higgs}})' \longrightarrow B' \quad \text{generic base.}$$

is an algebraic completely integrable system.

This system is often called Hitchin system.

- It provides important examples in mirror symmetry and geometric Langlands program.

$$\begin{array}{ccc} M^{\text{Higgs}}(G)' & & M^{\text{Higgs}}(G^L) \\ \downarrow & & \swarrow \\ B' & & \end{array}$$

"Roughly speaking", their fibers
are dual tori.

Proof for Part II

Step 1: Construct spectral curve for (E, ϕ) or
a pt in \mathbb{B} . $X(E, \phi)$

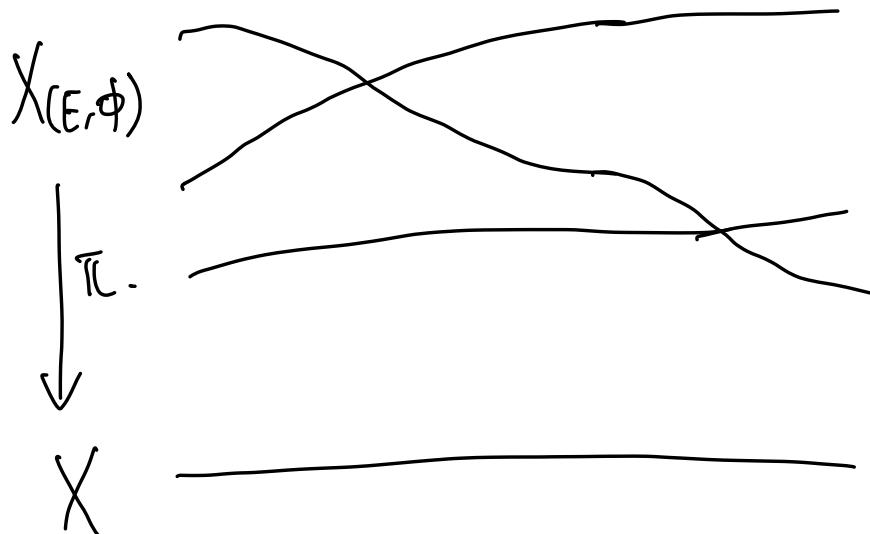
Given (E, ϕ) , define a spectral curve $\check{C} \subset \text{Tot}(K)$
 $\pi: K \rightarrow X$.

$$X_{(E,\phi)} := \{ \lambda \in \text{Tot}(K) \mid \det(\lambda \cdot \text{Id} - \phi) = 0 \}$$

Generically, each fiber of π has n pts.

We obtain an n -fold branched covering

$$\pi: X_{(E,\phi)} \rightarrow X .$$



Equivalently, let λ be the tautological 1-form on $\pi^* K$ over $\text{Tot}(K)$,
 $(\lambda(p) = p)$

then $X_{(E,\phi)}$ is the zero set of the polynomial

$$\begin{aligned} P_{(E,\phi)} &= \det(\lambda \text{Id} - \pi^*(\phi)) \\ &\in H^0(\text{Tot}(K), (\pi^* K)^n) \end{aligned}$$

We can also use another Hitchin fibration

$$\mathcal{H} = \mathcal{M}^{\text{Higgs}}(SL(n, \mathbb{C})) \rightarrow \mathcal{B}$$

$[(E, \phi)] \mapsto \text{coefficients of } \det(\lambda I - \phi).$

So given a $Q \in \mathcal{B}$, can define the spectral curve $X_Q := X_{(E, \phi)}$ for $(E, \phi) \in \mathcal{H}(Q)$.

Thm (Beauville - Narasimhan - Ramanan correspondence)

Assume Q in the Hitchin base is generic,

let H_Q be the Hitchin fiber at Q .

Then we have a biholom map between

$$(i) \quad H_Q \ni (\varepsilon, \phi) \longleftrightarrow \varrho \in \text{Pic}_0(X_Q)$$

(This is for $GL(n, \mathbb{C})$ -Higgs bundle case.)

(ii) Restrict to $SL(n, \mathbb{C})$ -Higgs bundles,

we have a biholom map between

$$H_Q \ni (\varepsilon, \phi) \longleftrightarrow \varrho \in \text{Prym}(X_Q \xrightarrow{\pi} X).$$

\cap

$$\text{Pic}_0(X_Q).$$

Rank: If ε has ϕ -inv subbundle, then

$$\det(\lambda I - \phi) = \det(\lambda I - \phi_1) \cdot \det(\lambda I - \phi_2)$$

for $\phi = \begin{pmatrix} \phi_1 & * \\ 0 & \phi_2 \end{pmatrix}$

So if $\det(\lambda I - \phi)$ is irreducible, then (Σ, ϕ) is stable.

Defn. Q is generic iff X_Q is generic,
 if X_Q is smooth and
 $\pi: X_Q \rightarrow X$ only has double
 ramification pt.
 $(\Rightarrow X_Q$ is connected.).

In particular, when $n=2$, $Q = (q_2)$,
 generic $\Leftrightarrow q_2$ has simple zeros.

In general, generic \Leftrightarrow locally,
 $P_Q(\lambda) = \prod_{i=1}^{n^2} (\lambda - \lambda_i(z)) \cdot (\lambda - \lambda_d(z))^2 - z$
 all λ_i are distinct.

\Leftrightarrow The discriminant Δ of P_Q has
 only simple zeros. $\Delta \in H^0(X, K^{n(n-1)})$.

Use Riemann-Hurwitz formula,

$$g_{X_Q} = g + (n^2 - 1)(g - 1).$$

Step 2: Construct two line bundles L, N on X_Q from (E, ϕ) .

W.r.t
 $\begin{cases} \chi(y dz) = y dz \\ \phi = (\chi(z)) dz \end{cases}$

Roughly, $N_y = \text{Ker}(\phi - y)$,

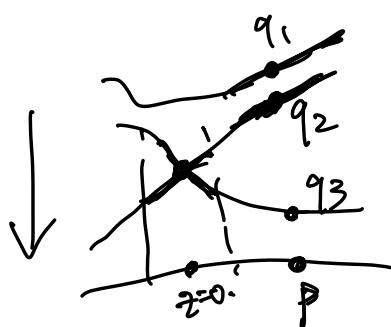
$$L_y = \text{Coker}(\phi - y)$$

$$\begin{cases} N = \text{Ker}(\pi^* \phi - \lambda) \subset \pi^* E \\ L = \text{Coker}(\pi^* \phi - \lambda) \text{ for } \pi^* E \end{cases}$$

- Away from branch pts.
- For $q_1, \dots, q_n \in X_Q$ above $p \in X$.

$$N_{q_2} = \text{Ker}(\phi - q_2).$$

i.e. if it is the eigen line bundle for q_2 .



$$L_{q_2} = \text{Coker}(\phi - q_2)$$

$$= E_{q_2} / \text{image}(\phi - q_2).$$

Note, $L_{q_2} \cong N_{q_2}$.

- Around branch pts:

Let's restrict to $N=2$ case.

The roots fail to be distinct precisely at $\text{Tr } \phi^2 = 0 = \det(\phi)$.

Pick a local complex coordinate z on X
s.t. $\text{Tr } \phi^2 = 2z dz^2$ near $z=0$.

Since we require $\text{Tr } \phi^2$ to have simple zeros,
the behavior of ϕ near $z=0$ is
conjugacy to $(\begin{smallmatrix} 0 & z \\ 1 & 0 \end{smallmatrix}) dz$ and $\phi(z) = (\begin{smallmatrix} 0 & z \\ 1 & 0 \end{smallmatrix})$.

The spectral curve near $z=0$ is given by

$$0 = \det(y - \phi) = y^2 - z.$$

$$0 = \det(\lambda - \phi) = (y^2 - z) dz^2$$

View y as a local parameter in X_Q .
 $y \mapsto y dz$

So at y , we want to solve

$$(\begin{smallmatrix} 0 & z=y^2 \\ 1 & 0 \end{smallmatrix}) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = y \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

So the soln is generated by

$$S = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} y \\ 1 \end{pmatrix} \text{ at } y.$$

$$\Rightarrow N_y = \mathbb{C} \cdot \begin{pmatrix} y \\ 1 \end{pmatrix}_S.$$

$\Rightarrow N_y = \text{Ker}(\varphi - y)$ is a well-defined line bundle over X_Q .

Next define $L_y = \text{Coker}(\varphi - y)$ near $z=0$.

$$\varphi - y = \begin{pmatrix} -y & z=y^2 \\ 1 & -y \end{pmatrix}$$

so the image of $\varphi - y$ is generated by $u = \begin{pmatrix} -y \\ 1 \end{pmatrix}$.

$$\Rightarrow \text{the quotient } L_y = E_y / \text{image}(\varphi - y) \\ = \mathbb{C} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{C}^2}.$$

$\Rightarrow L = \text{Coker}(\varphi - \lambda)$ is a well-defined line bundle over X_Q .

There is a natural holom map $N \rightarrow L$:

$$\theta: N \hookrightarrow \pi^* E \longrightarrow \pi^* E / \text{image}(\varphi - y) \\ L.$$

Away from branch pt, θ is an isom.

Near $z=0$, $\theta: N \rightarrow L$ is:

$$\begin{aligned}\theta\left(\begin{pmatrix} y \\ 1 \end{pmatrix}\right) &= \begin{pmatrix} y \\ 1 \end{pmatrix} \text{ mod } \mathbb{C} \cdot \begin{pmatrix} -y \\ 1 \end{pmatrix} \\ &\stackrel{s}{=} z^y \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{at } t.\end{aligned}$$

So a generator s of N is mapped to z^y . a generator t of L .

$\Rightarrow L = N(D)$ for D the ramification divisor for $X_Q \rightarrow X$.

So $L \cong N \otimes \underline{\pi^*(K^{N-1})}$. over(\phi, \lambda)

Step 3: Reconstruct (E, ϕ) from (L, X_Q) .

Roughly, $E = \pi^* L$.

$$\phi = \pi^*\lambda \quad \text{or} \quad \psi = \pi^*\gamma$$

for $\phi = \varphi(z)dz$, $\lambda = ydz$

Defn. If L is a line bundle on a branched covering $\tilde{X} \rightarrow X$ of n to 1, the push-forward $\pi_* L$ is the sheaf on X defined by:

over a sufficiently small open set $U \subset X$,

$$H^0(U, \pi_* L) := H^0(\pi^{-1}(U), L).$$

Prop. $\pi_* L$ is locally free, i.e. it defines a hol^m v.b of rk n .

Pf: Away from branch pt,

$$\pi^{-1}(U) = V_1 \cup \dots \cup V_n.$$

$$\begin{aligned} H^0(U, \pi_* L) &:= H^0(\pi^{-1}(U), L) \\ &= H^0(V_1, L) \oplus \dots \oplus H^0(V_n, L) \\ &\cong \Theta(V_1) \oplus \dots \oplus \Theta(V_n) \\ &\cong \Theta(U) \oplus \dots \oplus \Theta(U). \end{aligned}$$

Near branch pt,

$$\pi^{-1}(U) = V_1 \cup \dots \cup V_k \text{ of branch order } d_1, \dots, d_k,$$

$$\sum d_i = n.$$

$$V_R \rightarrow U$$

$$y \mapsto y^{d_i} = z.$$

$$\begin{aligned} H^0(U, \pi_* \mathcal{F}) &:= H^0(\pi^{-1}(U), \mathcal{F}) \\ &= H^0(V_1, \mathcal{F}) \oplus \dots \oplus H^0(V_k, \mathcal{F}) \\ &= \mathcal{O}(V_1) \oplus \dots \oplus \mathcal{O}(V_k). \end{aligned}$$

For each $\mathcal{O}(V_i)$, $\underline{\mathcal{O}(V_i)} \cong \{a_0(z) + a_1(z)y + \dots + a_{d_i-1}(z)y^{d_i-1}, a(z) \in \mathcal{O}(U)\}$.

$$\cong \mathcal{O}(U)^{d_i}.$$


About Prym variety question.

For a $\overset{\text{branch}}{\text{covering}}$ $\pi: S \rightarrow \Sigma$.

can the Norm map

$$Nm: \text{Pic}(S) \rightarrow \text{Pic}(\Sigma)$$

$$\sum n_i p_i \mapsto \sum n_i \pi(p_i).$$

$$\text{In particular, } Nm(\pi^{-1}(x)) = nx.$$

$\text{Ker}(Nm)$ is the Prym variety $\text{Prym}(S \rightarrow \Sigma)$
associated to $S \rightarrow \Sigma$.

$$\Lambda^n(\bar{\pi}_* \mathcal{L}) = 0 \quad (\text{for } \text{SL}(n, \mathbb{C})).$$

$$Nm(\mathcal{L}) \otimes K^{-\frac{n(n-1)}{2}}.$$

so $\Lambda^n(\bar{\pi}_* \mathcal{L})$ is trivial

$$\Leftrightarrow M := L \otimes \pi^* K^{-\frac{n-1}{2}} \in \text{Prym}(S, \Sigma).$$



Lecture 7

Goal: Finish step 3.

Show $E = \pi_* \mathcal{L}$ for $\mathcal{L} = \text{coker}(\pi^* \phi - \lambda)$
of $\pi^* E$

$$\phi = \pi_* \lambda.$$

Observe:

- A section of \mathcal{L} is the same thing as a section of $\pi^* E$ except we need to modulo $\text{Img}(\pi^*\phi - \lambda) \otimes \pi^{*K^{-1}}$
- A section of $\pi_k^*(\pi^* E)$ is the same thing as a section of E except that it allows the coefficient to depend on X_Q , not just on X .

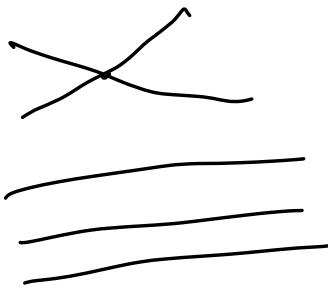
\Rightarrow a section of $\pi_k^*(\mathcal{L})$ is a section of E which

- (i) may depend on $y \in X_Q$.
- (ii) is considered trivial if it is in $\text{Img}(\pi^*\phi - \lambda) \otimes \pi^{*K^{-1}}$.

Prop. $\theta(\pi^*\mathcal{L})(U) \cong \theta(E)(U)$.

Pf: We only need to check
near branch pts.

Again, restrict to $n=2$ case.



- A section of E takes the form $\begin{pmatrix} a(z) \\ b(z) \end{pmatrix}$, with hol^m fns $a(z)$ $b(z)$
- A section of $\pi^*\mathcal{L}$ can be written

$$\begin{pmatrix} A(z) \\ B(z) \end{pmatrix} + y \begin{pmatrix} C(z) \\ D(z) \end{pmatrix} \quad \text{mod } \left(G(z) + y H(z) \right) \begin{pmatrix} -y \\ 1 \end{pmatrix}$$

$\pi^*(\pi^*E)$

Img $(\phi - \lambda) \otimes \mathbb{R}^2$.

$$\varphi = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} \quad y =$$

$$\varphi - y = \begin{pmatrix} -y & z = y^2 \\ 1 & -y \end{pmatrix}$$

Write as $\begin{bmatrix} A(z) + yC(z) \\ B(z) + yD(z) \end{bmatrix}$

Construct $f: \Theta(E)(U) \rightarrow \Theta(\bar{\Gamma}_k \ell)(U)$

$$\begin{pmatrix} a(z) \\ b(z) \end{pmatrix} \mapsto \begin{bmatrix} a(z) \\ b(z) \end{bmatrix}.$$

- injective

- surjective.

$$\begin{pmatrix} A(z) + yC(z) \\ B(z) + yD(z) \end{pmatrix} + (C(z) - yD(z)) \begin{pmatrix} -y \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} A(z) + zD(z) \\ B(z) + C(z) \end{pmatrix}.$$

$$\Rightarrow f\left(\begin{pmatrix} A(z) + zD(z) \\ B(z) + C(z) \end{pmatrix} \right) = \begin{bmatrix} A(z) + yC(z) \\ B(z) + yD(z) \end{bmatrix}$$

Continue to recover ϕ . ($\phi = \bar{\Gamma}_k \lambda$)

$$\text{Prop. } \mathcal{O}(E)(U) \xrightarrow{\phi = \varphi dz} \mathcal{O}(E)(U) \otimes \mathcal{O}(K)(U)$$

↓ ↗ ↓

$$\mathcal{O}(\tilde{\pi}_* \mathcal{F})(U) \xrightarrow{\begin{array}{l} \cdot \tilde{\pi}_*(\lambda) \\ \cdot \tilde{\pi}_*(y dz) \end{array}} \mathcal{O}(\tilde{\pi}_* \mathcal{F})(U) \otimes \mathcal{O}(K)(U)$$

Pf: A section f of \mathcal{F} is the same as
a section \hat{f} of $\tilde{\pi}_* E$ modulo $\text{Img}(\varphi - y)$

$$\text{Img}(\varphi - y) \otimes \tilde{\pi}_* K^*$$

We have $\varphi \hat{f} = y \hat{f} + (\varphi - y) \hat{f}$

$$\Rightarrow \varphi f = y f \text{ in } \mathcal{F}.$$

$$\Rightarrow \varphi \tilde{\pi}_* f = \tilde{\pi}_*(y f) \text{ in } \tilde{\pi}_* \mathcal{F}.$$

In summary: Given (E, ϕ) .



Define $L = \text{Coker}(\phi - \lambda)$

Note: $L = N(D)$ for $N = \ker(\phi - \lambda)$
 $D = \text{branch divisor of } x_Q \rightarrow x$.

Then we have

- $0 \rightarrow L(-D) \rightarrow \pi^* E \xrightarrow{\pi^*\phi - \lambda} \pi^*(E \otimes K)$
 ||
 N
- $E = \pi_* L$
- $\phi = \pi_* \lambda$.

In conclusion, we proved there is
a biholo^m between

$$\text{Pic}(X_Q) \longrightarrow H^{-1}(Q)$$

$$L \longmapsto (\pi_* L, \pi_* \lambda).$$

($\text{GL}(n, \mathbb{C})$ -case.)

Next question:

Given $L \in \text{Pic}(X_Q)$,

how much do we know about
 $(\pi_{*}L, \pi_{*}\lambda)$?

Two directions : • Properties for this map.
• Example : $L = \mathcal{O}_{X_Q}$.

Prop. Given an n -sheeted cover $S \rightarrow X$
 $\ell^0 \in \text{Pic}(X)$, $\ell \in \text{Pic}(S)$.

Then $\pi_{*}(\ell \otimes \pi^{*}\ell^0) = \pi_{*}\ell \otimes \ell^0$.

Pf: $\theta(\pi_{*}\ell)(U) \longleftrightarrow \theta(\ell)(\pi^{-1}U)$

$\theta(\pi_{*}(\ell \otimes \pi^{*}\ell^0))(U) \longleftrightarrow \theta(\ell \otimes \pi^{*}\ell^0)(\pi^{-1}U)$

St

$$\theta(\ell)(\pi^*(U)) \otimes \underline{\underline{\theta(\pi^*\ell^0)(U)}}^{\prime\prime}$$

//

$$\theta(\rho^0)(U)$$

$$\theta(\pi_*\ell)(U) \otimes \theta(\rho^0)(U).$$

□

$$\text{Prop. } \det(\pi_*\ell) \cong N_m(\ell) \otimes K^{-\frac{n(n-1)}{2}}.$$

Recall the norm map

$$N_m : \text{Pic}(S) \rightarrow \text{Pic}(\Sigma)$$

$$\sum n_i p_i \mapsto \sum n_i \pi(p_i).$$

$$\begin{aligned} \text{Therefore, } \det(\pi_*(\ell \otimes \pi^* K^{\frac{n-1}{2}})) \\ &= \det(\pi_*\ell \otimes K^{\frac{n-1}{2}}) \\ &= \det(\pi_*\ell) \otimes K^{\frac{n(n-1)}{2}} \\ &= N_m(\ell). \end{aligned}$$

Recall the Prym variety of $S \rightarrow \Sigma$ is

defined to be

$$\text{Prym}(S \rightarrow \Sigma) = \text{Kernel}(Nm).$$

- So for $SL(n, \mathbb{C})$ -Higgs bundle case,
we have a biholom φ for $Q \in \mathcal{B}^>_s(q_2, \dots, q_n)$

$$\begin{aligned}\text{Prym}(X_Q \rightarrow X) &\rightarrow H^1(Q) \\ \varphi &\mapsto (\pi^*(\varphi \otimes \pi^* K^{\frac{n-1}{2}}), \pi^*\lambda)\end{aligned}$$

From $\det(\pi^*(\varphi \otimes \pi^* K^{\frac{n-1}{2}}))$

$$= Nm(\varphi) = 0$$

Now, we give an example.

$$\text{Prop. } \pi_*(\mathcal{O}_{X_Q}) = \mathcal{O}_X \oplus K_X^{-1} \oplus K_X^{-2} \oplus \dots \oplus K_X^{1-n}$$

$$\pi_*(\lambda) = \begin{pmatrix} 0 & & & -q_n \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \vdots \\ & & & 1 & -q_2 \\ & & & & 1 & -q_1 \end{pmatrix}$$

$$: \pi_*(\mathcal{O}_{X_Q}) \rightarrow \pi_*(\mathcal{O}_{X_Q}) \otimes K_X$$

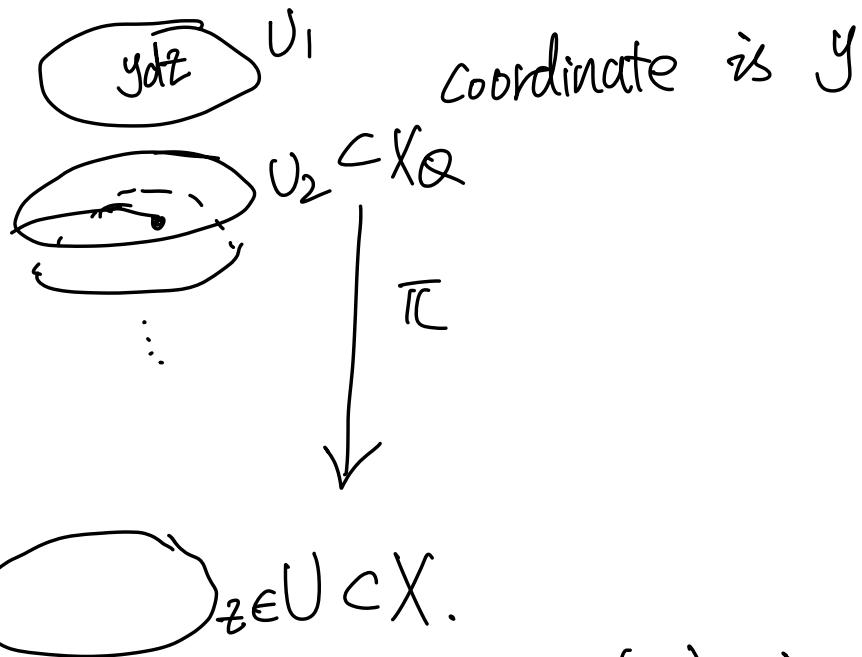
the companion matrix for
the polynomial

$$\det(\lambda - \phi) = \lambda^n + q_1 \lambda^{n-1} + q_2 \lambda^{n-2} + \dots + q_n.$$

for $\phi = (q_1, \dots, q_n) \in \mathbb{B}$.

Rmk: When $\text{tr} \phi = 0$, $q_1 = 0$.

Pf: Step 1: For $\pi_*(\mathcal{O}_{X_Q})$.



A local section of $\pi^*(\Omega_{X_Q})(U)$ is of
the form

$$\left\{ a_0(z) + a_1(z)y + \dots + a_{n-1}(z)y^{n-1} \right\}$$

for $a_i(z) \in \Omega_X(U)$.

If stops before y^n because

$$\begin{aligned} \det(\phi - \lambda) &= 0 & q_i(z) &= c_i(z) dz^i \\ \Rightarrow \det(\phi - y) &= 0 \\ \Rightarrow y^n + c_1(z)y^{n-1} + c_2(z)y^{n-2} + \dots + c_n(z) &= 0 \end{aligned}$$

We define the isomorphism between
 $\mathcal{O}_X(U)$ -modules

$$g_1: \pi_* (\mathcal{O}_{X_Q})(U_1) \longrightarrow \mathcal{O}_X(U_1)^n$$

$$a_0(z_1) + a_1(z_1)y_1 + \dots + a_{n-1}(z_1)y_1^{n-1} \mapsto \begin{pmatrix} a_0(z_1) \\ a_1(z_1) \\ \vdots \\ a_{n-1}(z_1) \end{pmatrix}$$

On (U_1, z_1) , we have g_1 .
 $(\pi^{-1}U_1, y_1)$

On (U_2, z_2) , we have g_2

$$g_2: \pi_* (\mathcal{O}_{X_Q})(U_2) \longrightarrow \mathcal{O}_X(U_2)^n$$

$$b_0(z_2) + b_1(z_2)y_2 + \dots + b_{n-1}(z_2)y_2^{n-1} \mapsto \begin{pmatrix} b_0(z_2) \\ b_1(z_2) \\ \vdots \\ b_{n-1}(z_2) \end{pmatrix}$$

So on $U_1 \cap U_2$,

$$g_{12} = g_2 \circ g_1^{-1}: \mathcal{O}_X(U_1 \cap U_2)^n \longrightarrow \mathcal{O}_X(U_1 \cap U_2)^n.$$

From the coordinate choice of y ,

$$y_1 dz_1 = y_2 dz_2 \implies \frac{y_1}{y_2} = \frac{dz_2}{dz_1}.$$

From \mathcal{O}_{X_Q} is trivial,

$$a_0(z_1) + a_1(z_1)y_1 + \dots + a_{n-1}(z_1)y_1^{n-1}$$

$$= b_0(z_2) + b_1(z_2)y_2 + \dots + b_{n-1}(z_2)y_2^{n-1}.$$

$$\Rightarrow \left\{ \begin{array}{l} b_0(z_2) = a_0(z_1) \\ b_1(z_2) = a_1(z_1) \cdot \frac{y_1}{y_2} = a_1(z_1) \cdot \frac{dz_2}{dz_1} \\ \dots \\ b_{n-1}(z_2) = a_{n-1}(z_1) \cdot \left(\frac{y_1}{y_2} \right)^{n-1} = a_{n-1}(z_1) \cdot \left(\frac{dz_2}{dz_1} \right)^{n-1} \end{array} \right.$$

$$\Rightarrow g_{12} = \begin{pmatrix} 1 & & & \\ & \frac{dz_2}{dz_1} & & \\ & & \ddots & \\ & & & \left(\frac{dz_2}{dz_1} \right)^{n-1} \end{pmatrix}$$

So the rk n v.b glued from such transition fns

$$\text{is } \Omega_X \oplus K_X^{-1} \oplus \dots \oplus K_X^{1-n}$$

$$\text{In particular, } \Omega(K_X^{-i})(U) \cong \{ a(z) y^i \} \cong f_i(U).$$

$a(z) \frac{dy}{y^i} \mapsto a(z) y^i.$

Step 2: For $\pi_* \lambda$.

For $0 \leq i \leq n-2$.

$$\begin{array}{ccc}
 T_{(\ast)\lambda}: F_i(U) & \xrightarrow{\cdot y dz} & \tilde{F}_{i+1}(U) \otimes \theta(K_X)(U) \\
 \downarrow a(z)y^i & \downarrow \xrightarrow{\cdot y dz} & \downarrow \cancel{a(z)y^{i+1}} dz \\
 & & a(z)^{-i-1} \cdot dz \\
 \downarrow a(z)dz^{-i} & \xrightarrow{I} & \downarrow a(z)dz^{-i-1} \cdot dz \\
 I: \theta(K_X^{-i})(U) & \longrightarrow & \theta(K_X^{-i-1}(U)) \otimes \theta(K_X)(U)
 \end{array}$$

For $i=n-1$.

$$\begin{array}{ccc}
 T_{(\ast)\lambda}: F_{n-1}(U) & \xrightarrow{\cdot y dz} & (\tilde{F}_0 \oplus \dots \oplus \tilde{F}_{n-1})(U) \otimes \theta(K_X)(U) \\
 \downarrow a(z)y^{n-1} & \longrightarrow & a(z)y^n dz \\
 & & a(z)(-c_1(z)y^{n-1} - c_2(z)y^{n-2} - \dots - c_n(z)) dz \\
 \downarrow a(z)dz^{1-n} & \longrightarrow & (a(z)c_1(z), -a(z)c_2(z)dz^{-1}, \dots, -a(z)c_n(z)dz^{1-n}) \\
 (-q_n, \dots, q_1): \theta(K_X^{1-n})(U) & \longrightarrow & (\theta \oplus \theta_U(K_X^{-1}) \oplus \dots \oplus \theta_U(K_X^{1-n})) \otimes \theta_U(K)
 \end{array}$$

Combine together,
we obtain $T_{(\ast)\lambda} = \begin{pmatrix} 0 & -q_1 \\ \vdots & \ddots & \vdots \\ 0 & \ddots & -q_2 \\ & \ddots & \ddots & \vdots \\ & & \ddots & -q_n \end{pmatrix}$. □

$$\text{Cor. } \pi_*(\pi^* K^{\frac{n-1}{2}}) = \pi_*(\mathcal{O}_X) \otimes K^{\frac{n-1}{2}}$$

$$= K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \dots \oplus K^{\frac{1-n}{2}}.$$

$$\pi_* \lambda = \begin{pmatrix} 0 & -q_n \\ \vdots & \vdots \\ \ddots & -q_2 \\ \ddots & \ddots & -q_1 \\ 1 & -q_1 \end{pmatrix}$$

This gives a section for $H: (\mathcal{M}^{\text{Higgs}})' \rightarrow \mathcal{B}'$

In fact, we can extend this section to \mathcal{B} .

$$\text{Given } (q_1, \dots, q_n) \in \mathcal{B},$$

$$s(q_1, \dots, q_n) = \begin{pmatrix} K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \dots \oplus K^{\frac{1-n}{2}} \\ \left(\begin{array}{cccc} 0 & 0 & \dots & -q_n \\ \vdots & \vdots & \ddots & -q_2 \\ 1 & -q_1 \end{array} \right) \end{pmatrix}$$

called Hitchin section.

Rmk: • If $q_i = 0$, $s(0, q_2, \dots, q_n) \in \mathcal{M}^{\text{Higgs}}(\text{SL}(n, \mathbb{C}))$.

• $\text{Img}(s)$ is a closed subspace of $\mathcal{M}^{\text{Higgs}}$.

• How about general Lie groups?

For this, Hitchin uses Kostant's principle

3-dim subgroup theory.

Even for $SL(n, \mathbb{C})$ case, the canonical expression
(from Lie algebra) is

$$E = K^{\frac{1-n}{2}} \oplus \dots \oplus K^{\frac{n-1}{2}} \quad (Q = \begin{pmatrix} & & & 1 \\ & \ddots & & \\ & & \ddots & \\ 1 & & & \end{pmatrix})$$

$$\phi = \begin{pmatrix} 0 & q_2 & q_3 & \cdots & q_n \\ r_1 & 0 & & & \\ r_2 & & 0 & & \\ & \ddots & & \ddots & q_3 \\ & & & & 0 \\ & & & & r_{n-1} & 0 \end{pmatrix} \quad \text{for } r_i = \frac{i(n-i)}{2}.$$

which is conjugate to the companion matrix.

- One can show that Hitchin section only contains stable Higgs bundles.
Similar to the case for $(K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix})$.

(i) Show $q_2 = 0, \dots, q_n = 0$,

the ϕ -inv subbundles have $\deg < 0$.

(ii) Use the openness of stability.

(iii) Use \mathbb{C}^* -action.

$$t \cdot s(q_2, \dots, q_n) = s(t^2 q_2, \dots, t^n q_n). \quad \blacksquare$$

- Because of stability, $\nabla s(q_2, \dots, q_n)$,
 \exists a harmonic metric H

\Rightarrow obtain a flat connection D on E .

In fact, Hitchin showed that
 $\text{Hol}(D) \in \text{Hom}(\pi, \text{SL}(n, \mathbb{R}))$
up to conjugate.