

Lecture 8

Harmonic flat vector bundles

From previous lectures, we know that

polystable Higgs bundles (algebraic condition)

\Leftrightarrow harmonic Higgs bundles $(E, \bar{\partial}_E, \phi, H)$

→ flat vector bundles (E, D) harmonic,
 i.e. satisfying
 by $D = \nabla^H + \phi + \phi^* H$ Hitchin eqn

Question: How to go back?

If is a triple (E, D, H) .

$$= \sqrt{1-i} + (\text{左}\sum H)^{1,0} + (\text{右}\sum H)^{0,1}.$$

Candidate for a Higgs bundle is

$$(\mathcal{F}^H)^{0,1} \quad (\mathcal{F}^H)^{1,0})$$

holo str.

$$(Z^{\ast} - H)^{1,0})$$

$$\mathbb{A}^{l,0}(X, \text{End}(E)).$$

Want to give an appropriate condition on H
to make sure this pair
is indeed a Higgs bundle.

Answer: Harmonic metric.

§0. Preparation.

On a R.S X , fix a background conformal metric
 $g_0 = g_0(z) (dz \otimes d\bar{z} + d\bar{z} \otimes dz)$

with volume form $dvol_{g_0} = i g_0(z) dz \wedge d\bar{z}$.

- The Hodge star $*$ is a conjugate-linear map
 $*: A^k(X, \mathbb{C}) \rightarrow A^{2-k}(X, \mathbb{C})$

$$\begin{cases} *dx = dy \\ *dy = -dx \end{cases}$$

s.t. $\alpha_1 \wedge * \alpha_2 = \langle \alpha_1, \alpha_2 \rangle_{g_0} dvol_{g_0}$.

In particular, $\begin{cases} *dz = id\bar{z} \\ *\bar{z} = -idz \end{cases}$ Also define $*_{\mathbb{C}}$ \mathbb{C} -linear

$$\begin{cases} *\bar{z} = -idz \\ *\bar{z} = id\bar{z} \end{cases}$$

- Let (E, h) be a Hermitian v.b.

On $A^k(X, \text{End}(E))$, one can define the Hodge

star operator $*_h, *$ as follows:

for $\Psi = \sum_{i=1}^l \alpha_i \otimes \psi_i \in A^k(X, \text{End}(E))$,
 $\Psi \in A^k(X, \mathbb{C})$

$$\left\{ \begin{array}{l} *_{h\#} \varphi := \sum_{i=1}^l *_{\varphi_i} \otimes \varphi_i^{*h} \\ *_{\#} \varphi := \sum_{i=1}^l *_{\varphi_i} \otimes \varphi_i \end{array} \right. \quad \text{the adjoint of } \varphi_i. \quad \text{well-defined.}$$

Note that $*_{h\#} = *_{h(\#)}^{1,0} + *_{h\#}^{0,1}$

$$= i(\#^{1,0})^{*h} - i(\#^{0,1})^{*h}$$

When $\#$ is self-adjoint, $\#^{*h} = \#$.

φ_i Hermitian $\Rightarrow (\#^{1,0})^{*h} + (\#^{0,1})^{*h} = \#^{1,0} + \#^{0,1}$

So
$$*_{h\#} = i\#^{0,1} - i\#^{1,0} = *_{\#}.$$

(For $\#$ self-adjoint, $\# = \phi dz + \phi^{*h} d\bar{z}$
 $*_{\#} = -i\phi dz + i\phi^{*h} d\bar{z} = i\#^{0,1} - i\#^{1,0}$)

- On $A^k(X, \text{End}(E))$, one can define a L^2 pairing as follows:

$$\langle \#, \#_2 \rangle_{L^2} := \int_X \text{tr}(\#_1 \wedge *_{h\#} \#_2).$$

§1. Harmonic metric on a flat v.b (E, D)

Defn. The energy of a Hermitian metric H on (E, D) is defined as

$$E(H) := \left\| \mathbb{P}_H \right\|_{L^2}^2 = \int_X \text{tr}(\mathbb{P}_H \wedge * \mathbb{P}_H)$$
$$(D = \nabla^H + \mathbb{P}_H) \quad (= \int_X \text{tr}(\mathbb{P}_H \wedge * \mathbb{P}_H))$$

Defn. H is called harmonic

if it is a critical pt of the energy functional $E(H)$ among all Hermitian metrics.

Prop. H is harmonic $\Leftrightarrow \nabla^H * \mathbb{P}_H = 0$.

Pf: First, change viewpoint.
If we start with a triple (E, D_0, H_0) ,
we want to find $H = g \cdot H_0$ s.t
it is a critical pt of $\left\| \mathbb{P}_{(D_0, g \cdot H_0)} \right\|_{L^2, g \cdot H_0}^2$

$$\left\| \mathbb{P}_{(g^{-1} \cdot D_0, H_0)} \right\|_{L^2, H_0}^2$$

So the question can be changed into:

Fix (E, H_0) , look for $D \in G \cdot D_0$
s.t. it is a critical pt of $\|\Psi_{g \cdot D_0, H_0}\|_{L^2, H_0}^2$

Then it is enough to characterize the
critical pt D for $\|\Psi_{g \cdot D}\|_{L^2, H_0}^2$
in the gauge orbit of D

as $\nabla^{H_0} * \Psi_D = 0.$

where $D = \nabla^{H_0} + \Psi_D$.

Note that $g \cdot D = g \circ D \circ g^{-1}$.

$$\begin{aligned} \text{So } (g \circ D \circ g^{-1})(s) &= g(D(g^{-1}s)) \\ &= g(g^{-1}Ds + Dg^{-1} \cdot s) \\ &= DS + gDg^{-1}s \\ &= DS - Dg \cdot g^{-1}s. \end{aligned}$$

(using $Dg^{-1} = -g^{-1}Dg g^{-1}$.)

$$\Rightarrow g \cdot D = D - Dg \cdot g^{-1}$$

Lemma. $\nabla_{g \cdot D} = \nabla_D - \frac{1}{2} Dg \cdot g^{-1} - \frac{1}{2} g^*, -1 \nabla_D g^*$

where $\hat{D} = \nabla^{H_0} - \nabla_D$.

Pf of Lemma. It is enough to show

$$(Dg \cdot g^{-1})^* = g^*, -1 \hat{D} g^*.$$

First we have a formula

$$\langle DS, t \rangle + \langle S, \hat{D} t \rangle = d\langle S, t \rangle.$$

$$\begin{aligned} & \underbrace{\langle DS, t \rangle}_{\text{II}} + \underbrace{\langle S, \hat{D} t \rangle}_{d\langle S, t \rangle} = \underbrace{d\langle S, t \rangle}_{\text{II}} \\ & \underbrace{\langle \nabla^{H_0} S, t \rangle}_{\text{II}} + \underbrace{\langle S, \nabla^{H_0} t \rangle}_{d\langle S, t \rangle} + \underbrace{\langle \nabla_D S, t \rangle - \langle S, \nabla_D t \rangle}_{\text{II}} \end{aligned}$$

$$\begin{aligned} & \langle S, Dg \cdot g^{-1} t \rangle \\ &= \langle S, D(g \cdot g^{-1} t) \rangle - \langle S, g \cdot Dg^{-1} t \rangle \\ &= \langle S, D t \rangle - \underbrace{\langle g^* S, Dg^{-1} t \rangle}_{d\langle S, t \rangle} \\ &= \langle S, D t \rangle - \underbrace{d\langle g^* S, g^{-1} t \rangle}_{d\langle S, t \rangle} + \langle \hat{D}(g^* S), g^{-1} t \rangle \\ &= -\langle DS, t \rangle + \langle \hat{D}(g^* S), g^{-1} t \rangle \end{aligned}$$

$$\begin{aligned}
&= -\langle g^* \tilde{D}S, \tilde{g}^t \rangle + \langle \tilde{D}(g^* S), \tilde{g}^t \rangle \\
&= \langle (\tilde{D}g^*)S, \tilde{g}^t \rangle \\
&= \langle g^{*-1} \tilde{D}g^* S, t \rangle. \quad \square
\end{aligned}$$

Return to the characterization of harmonicity.

For $g = \exp(t\beta)$, $\beta \in \mathcal{A}^0(X, \text{End}(E))$.

$$\begin{aligned}
\text{Then } d(E(g \cdot D))_{g=1}(\beta) &= d \left\langle \tilde{\int}_D g \cdot D, \tilde{\int}_D \right\rangle_{L^2, g=1}(\beta) \\
&= 2 \operatorname{Re} \left\langle d \tilde{\int}_D(\beta), \tilde{\int}_D \right\rangle_{L^2, g=1}. \\
&= 2 \operatorname{Re} \left\langle d \left(\tilde{\int}_D -\frac{1}{2} Dg \cdot g^{-1} - \frac{1}{2} g^* \cdot {}^{-1}\tilde{D}g^* \right) \beta, \tilde{\int}_D \right\rangle_{L^2, g=1} \\
&= 2 \operatorname{Re} \left\langle d \left(\tilde{\int}_D -\frac{1}{2} D\beta - \frac{1}{2} \tilde{D}\beta^* \right), \tilde{\int}_D \right\rangle_{L^2, g=1} \\
\text{Apply } D = \nabla^{H_0} + \tilde{\int}_D, \quad \tilde{D} = \nabla^{H_0} - \tilde{\int}_D &= 2 \operatorname{Re} \left\langle -\frac{1}{2} D\beta - \frac{1}{2} \tilde{D}\beta^*, \tilde{\int}_D \right\rangle_{L^2} \\
&= -\operatorname{Re} \left\langle \nabla^{H_0} \beta + [\tilde{\int}_D, \beta] + \nabla^{H_0} \beta^* - [\tilde{\int}_D, \beta^*], \tilde{\int}_D \right\rangle_{L^2}
\end{aligned}$$

$$= -\operatorname{Re} \langle \nabla^{H_0}(\xi + \xi^*) + [\mathbb{I}_D, \xi - \xi^*], \mathbb{I}_D \rangle_{L^2}$$

Lemma 1. $\langle [\mathbb{I}_D, X], \mathbb{I}_D \rangle_{L^2} = 0$. (will be proved later)

Assuming Lemma 1.

$$= -\operatorname{Re} \langle \nabla^{H_0}(\xi + \xi^*), \mathbb{I}_D \rangle_{L^2}$$

$$= -\operatorname{Re} \operatorname{str}(\nabla^{H_0}(\xi + \xi^*) \wedge * \mathbb{I}_D)$$

$$= \underbrace{\operatorname{Re} \operatorname{tr}((\xi + \xi^*) \cdot \nabla^{H_0} * \mathbb{I}_D)}_{=0} + \underbrace{\operatorname{Re} \operatorname{tr}(\nabla^{H_0}((\xi + \xi^*) \wedge * \mathbb{I}_D))}_{\text{because } \operatorname{tr}(\nabla^{H_0}((\xi + \xi^*) \wedge * \mathbb{I}_D)) \text{ is exact.}} \\ \text{see Lemma 2 on next page.}$$

$$= \operatorname{Re} \langle \xi + \xi^*, *^{-1}_h \nabla^{H_0} * \mathbb{I}_D \rangle.$$

Since ξ is arbitrary, and $*^{-1}_h \nabla^{H_0} * \mathbb{I}_D$ is self-adjoint,

then D is a critical pt.

$$\Leftrightarrow *^{-1}_h \nabla^{H_0} * \mathbb{I}_D = 0$$

$$\Leftrightarrow \nabla^{H_0} * \mathbb{I}_D = 0. \quad \blacksquare$$

Rmk: $D^{H_0} * \mathbb{I}_D = 0 \Leftrightarrow (D^{H_0})^* \mathbb{I}_D = 0$,

where $(D^{H_0})^* = -*^{-1}_h D^{H_0} *_h$.

Lemma 1: $\langle [\Psi, X], \Psi \rangle_{L^2} = 0$ for $X \in \mathcal{A}^0(X, \text{End}(E))$
Pf of Lemma: $\Psi \in \mathcal{A}^1(X, \text{End}(E))$
and Ψ self-adjoint.

$$\begin{aligned}
& \langle [\Psi, X], \Psi \rangle_{L^2} \\
&= \int \text{tr}([\Psi, X] \wedge * \Psi) \\
&= \int \text{tr}((\Psi X - X \Psi) \wedge * \Psi) \\
&= \int \text{tr}(-X * \Psi \wedge \Psi - X \Psi \wedge * \Psi) \\
&= - \int \text{tr}(X \cdot [\Psi, * \Psi]) = - \langle X, *_h^{-1}[\Psi, * \Psi] \rangle.
\end{aligned}$$

Claim: $[\Psi, * \Psi] = 0$.

$$\begin{aligned}
\text{Pf: } [\Psi, * \Psi] &= [\Psi^{1,0} + \Psi^{0,1}, i\Psi^{0,1} - i\Psi^{1,0}] \\
&= i[\Psi^{1,0}, \Psi^{0,1}] - i[\Psi^{0,1}, \Psi^{1,0}] = 0. \quad \square
\end{aligned}$$

Lemma 2: $\text{tr}(\nabla \xi)$ is exact, for $\xi \in \mathcal{A}^k(X, \text{End}(E))$.

Pf of Lemma: Locally, $\nabla = d + A$.

$$\begin{aligned}
\text{then } \text{tr}(\nabla \xi) &= \text{tr}(d\xi + [A, \xi]) \\
&= \text{tr}(d\xi) \\
&= d \text{tr}(\xi).
\end{aligned}$$

Since $d \text{tr}(\xi)$ is globally well-defined,

$$\text{tr}(\nabla \xi) = d \text{tr}(\xi). \quad \square$$

§2. Harmonic flat bundles VS harmonic Higgs bundles.

Defn. A harmonic flat bundle $\Rightarrow (E, D, H)$
 \uparrow flat \uparrow harmonic

Defn. A harmonic Higgs bundle $\Rightarrow (\underbrace{E, \bar{\partial}E}_{\text{Higgs bundle}}, \phi, H)$
 \uparrow Higgs bundle \uparrow harmonic

Lemma. \exists a natural bijection between

$$\{\text{harmonic flat bundles}\} \rightarrow \{\text{harmonic Higgs bundles}\}$$

$$(E, D, H) \mapsto (E, \nabla_H^{0,1}, \bar{\nabla}_H^{1,0}, H)$$

$$(E, \nabla_H^H + \phi + \phi^*, H) \leftarrow (E, \bar{\partial}E, \phi, H)$$

Pf: (E, D, H) is flat harmonic $\Rightarrow \begin{cases} F_D = 0 & \text{(flatness)} \\ \nabla^H * \bar{\nabla}_H = 0 & \text{(harmonicity)} \end{cases}$

$$\Rightarrow \begin{cases} F_D = 0 & \text{(flatness)} \\ \nabla^H * \bar{\nabla}_H = 0 & \text{(harmonicity)} \end{cases} \quad \begin{matrix} \text{(1)} \\ \text{(2)} \end{matrix}$$

$$\begin{matrix} \text{(unitary)} \\ \text{(Hermitian)} \end{matrix}$$

Claim: Eqn (1) \Leftrightarrow Hitchin eqn.
 $\bar{\nabla}_H \wedge \bar{\nabla}_H = (\bar{\nabla}_H^{1,0} + \bar{\nabla}_H^{0,1}) \wedge (\bar{\nabla}_H^{1,0} + \bar{\nabla}_H^{0,1})$
 $= [\bar{\nabla}_H^{1,0}, \bar{\nabla}_H^{0,1}]$.

Note that $\vec{\Psi}_H^{0,1} = (\vec{\Psi}_H^{1,0})^*_{\text{h}}$
because $\vec{\Psi}_H^*_{\text{h}} = \vec{\Psi}_H$.

Claim: Eqn (2) + (3) \iff hole of Higgs field
i.e. $(\nabla^H)^{0,1} \vec{\Psi}_H^{1,0} = 0$.

Since $*\vec{\Psi}_H = i\vec{\Psi}_H^{1,0} - i\vec{\Psi}_H^{0,1}$
Eqn (2) + (3) are $\begin{cases} \nabla^H * \vec{\Psi}_H = i(\nabla^H \vec{\Psi}_H^{1,0} - \nabla^H \vec{\Psi}_H^{0,1}) = 0 \\ \nabla^H \vec{\Psi}_H = \nabla^H \vec{\Psi}_H^{1,0} + \nabla^H \vec{\Psi}_H^{0,1} = 0 \end{cases}$

$$\iff \begin{cases} \nabla^H \vec{\Psi}_H^{1,0} = 0 \\ \nabla^H \vec{\Psi}_H^{0,1} = 0 \end{cases}$$

$$\iff \begin{cases} (\nabla^H)^{0,1} \vec{\Psi}_H^{1,0} = 0 \\ (\nabla^H)^{1,0} \vec{\Psi}_H^{0,1} = 0 \end{cases}$$

$$\iff (\nabla^H)^{0,1} \vec{\Psi}_H^{1,0} = 0. \quad \square$$

Summarize everything,

polystable Higgs bundles

\iff harmonic Higgs bundles

\iff harmonic flat bundles

\iff **reductive** flat bundles

Defn. Let (E, D) be a flat v.b.

Call D **reductive** if any D -inv subbundle
(completely reducible)
has a D -inv complement.

Call D **irreducible** if \exists no proper
 D -inv subbundle.

Ihm (Corlette, Donaldson)

On a flat v.b (E, D) ,

it is reductive iff \exists a harmonic metric.

Moreover, if it is irreducible, the harmonic
metric is unique up to constant scalar.

" \Rightarrow " hard. (We won't show it.)

Pf: " \Leftarrow " easy. Let's show this direction.

Given a harmonic flat v.b (E, D, H) ,

let E_1 be a D -inv subbundle of E .

Take $E_2 = E_1^{\perp_H}$.

W.r.t $E = E_1 \oplus E_2$,

$$H = \begin{pmatrix} H_1 & \\ & H_2 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & \Phi \\ 0 & D_2 \end{pmatrix}$$

Suppose $\Phi \neq 0$, we will show ∇ by
saying H is not a critical pt of $E(H)$.

Consider a family of Herm metrics H_t ,

$$H_t = \begin{pmatrix} t^{rk E_2} H_1 & \\ & t^{-rk E_1} H_2 \end{pmatrix}$$

W.r.t H_t , $D = \nabla_{H_t} + \frac{1}{2}\Phi_{H_t}$

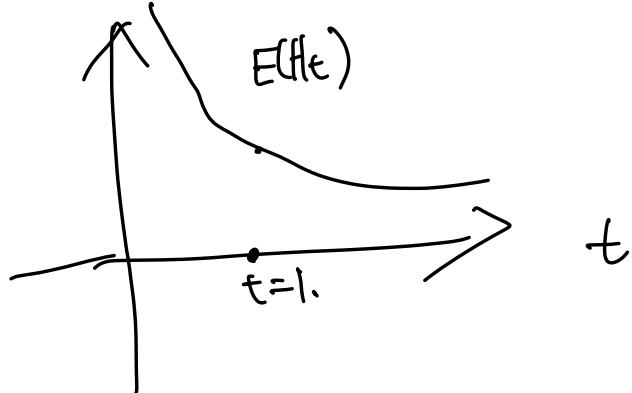
$$= \begin{pmatrix} \nabla_{H_1} & \frac{1}{2}\Phi \\ -\frac{1}{2}\Phi^*_{H_t} & \nabla_{H_2} \end{pmatrix} + \begin{pmatrix} \Phi_{H_1} & \frac{1}{2}\Phi \\ \frac{1}{2}\Phi^*_{H_t} & \Phi_{H_2} \end{pmatrix}$$

$$\begin{aligned}
\text{So } E(H_t) &= \|\Psi_{H_t}\|_{L^2, H_t}^2 \\
&= \int \text{tr} \left(\begin{pmatrix} \Psi_{H_1} & \frac{1}{2} \Phi \\ \frac{1}{2} \Phi^*_{H_t} & \Psi_{H_2} \end{pmatrix} \wedge * \begin{pmatrix} \Psi_{H_1} & \frac{1}{2} \Phi \\ \frac{1}{2} \Phi^*_{H_t} & \Psi_{H_2} \end{pmatrix} \right) \\
&= \|\Psi_{H_1}\|_{L^2, H_1}^2 + \frac{1}{4} \int \text{tr} (\Phi \wedge * \Phi^*_{H_t}) \\
&\quad + \frac{1}{4} \int \text{tr} (\Phi^*_{H_t} \wedge * \Phi) + \|\Psi_{H_2}\|_{L^2, H_2}^2
\end{aligned}$$

$$= \|\Psi_{H_1}\|_{L^2, H_1}^2 + \|\Psi_{H_2}\|_{L^2, H_2}^2 + \frac{1}{2} \int \text{tr} (\Phi \wedge * \Phi^*_{H_t})$$

$$\text{Claim: } \Phi^*_{H_t} = t^{-rKE} \Phi^*_h$$

$$\begin{aligned}
\text{Assuming } \text{Claim} \\
= \|\Psi_{H_1}\|_{L^2, H_1}^2 + \|\Psi_{H_2}\|_{L^2, H_2}^2 + t^{-rKE} \int \text{tr} (\Phi \wedge * \Phi^*_h)
\end{aligned}$$



So H cannot be a critical pt. 

Next lecture:

- Relation between harmonic metrics and harmonic maps.
 - Finish talking about NAFI.

Lecture 9

§1. ρ -equivariant harmonic maps

Let (S, g_0) , (N, h) Riemannian mfld.

Defn. Given a rep $\rho: \pi_1(S) \rightarrow \text{Isom}(N)$,

a map $f: \tilde{S} \rightarrow N$ is called ρ -equiv

if $f(\sigma \cdot x) = \rho(\sigma) \cdot f(x) \quad \forall \sigma \in \pi_1(S), x \in \tilde{S}$.

For a ρ -equiv map $f: (\tilde{S}, \tilde{g}_0) \rightarrow (N, h)$,

$df \in \Gamma(\tilde{S}, T^*\tilde{S} \otimes f^*TN)$ is also ρ -equiv.

So the fcn $e(f) := \frac{1}{2} \frac{\|df\|^2}{\|df\|^2_{T^*\tilde{S} \otimes f^*TN}}$

is also ρ -equiv.

Can write

$$\underbrace{\langle df, df \rangle}_{\|df\|^2} d\text{vol}_{\tilde{g}_0} = g_{f^*TN} (df \wedge *df).$$

Here $*: \mathcal{A}^1(\tilde{S}, \mathbb{R}) \rightarrow \mathcal{A}^1(\tilde{S}, \mathbb{R})$

$$*dx = dy$$

$$*dy = -dx$$

hence descends to a fcn on S .

- Call $e(f)$ the energy density on S .

- The energy $E(f) := \int_S \text{eff}) d\text{vol}_{g_0}$
 $= \int_S g_{f^*TN} (df \wedge *df).$

Note that $E(f)$ is finite if S is cpt.

Defn. A ρ -equiv map $f: \tilde{S} \rightarrow N$ is called harmonic if it is a critical pt of the energy functional $E(f)$ among all ρ -equiv maps.

Equivalently, f is harmonic $\Leftrightarrow \text{tr}_{g_0} \underline{\nabla df} = 0 \in T^*S \otimes T^*S \otimes f^*TN$
 where ∇ is the connection on

$T^*S \otimes f^*TN$ induced by Levi-Civita conn.

Rmk: Note that $\text{eff})$ depends on the metric g_0 ,
 but $E(f)$ only depends on the conformal class of g_0 .

So we can talk about harmonic maps
 from a Riemann surface
 instead of a Riemannian surface.

Let N be the space of pos. def $n \times n$ Hermitian matrices.

$GL(n, \mathbb{C})$ acts on N by $g \cdot A = (g^{-1})^* A g^{-1}$.

Note that $\text{Stab}_{\text{Id}}(GL(n, \mathbb{C})) = U(n)$.

$$N \cong GL(n, \mathbb{C}) / U(n).$$

The Riemannian metric g_N on N is defined as follows:

At $A \in N$, $T_A N \cong \{ \text{Hermitian } n \times n \text{ matrices} \}$

$$\forall X, Y \in T_A N$$

$$g_N(X, Y)|_A := \text{tr}(A^{-1}X A^{-1}Y).$$

One can check that the metric g_N is $GL(n, \mathbb{C})$ -inv.

§2. Equivalence between harmonic metric and ρ -equiv harmonic maps

Step 1: Metric H on $(E, D) \Leftrightarrow$ ρ -equiv map $f: \tilde{X} \xrightarrow{\cong} N$
 flat $\Updownarrow_{H^0(D)}$ $\begin{matrix} \text{flat} \\ \text{flat} \end{matrix}$ $\begin{matrix} \text{a connection} \\ \text{descends} \\ \text{from } d \\ \text{on } \tilde{X} \times \mathbb{C}^n \end{matrix}$

First, identify

$$(E, D) \cong (\tilde{X} \times_{\rho} \mathbb{C}^n := \tilde{X} \times \mathbb{C}^n / \sim, \quad \begin{matrix} \text{a connection} \\ \text{descends} \\ \text{from } d \\ \text{on } \tilde{X} \times \mathbb{C}^n \end{matrix})$$

$$(\tilde{x}, v) \sim (\gamma \cdot \tilde{x}, \rho(\gamma) \cdot v).$$

Lift H to \tilde{H} on $\tilde{X} \times \mathbb{C}^n$, $\tilde{D} = d$.

Define $f_H^{(x)} = (\tilde{H}_x(e_i, e_j))_{ij}$ for $\{e_i\}$ standard basis of \mathbb{C}^n .

Claim: $f_H(x) = \overline{\rho(\gamma)^T f_H(\gamma x)} \rho(\gamma)$.

If: Since \tilde{H} is a lift from E ,

$$(x, v) \sim (\gamma x, \rho(\gamma)v).$$

$$\text{so } \tilde{H}(x)(v, w) = \tilde{H}(\gamma x)(\rho(\gamma)v, \rho(\gamma)w)$$

$$\Rightarrow \nabla^T \underline{f_H(x)} w = \overline{\rho(\gamma)v^T f_H(\gamma x) \rho(\gamma)w}.$$

$$= \overline{\nabla^T \underline{\rho(\gamma)^T f_H(\gamma x) \rho(\gamma)w}}. \quad \square$$

Conversely, given f , define H_f on $\tilde{X} \times \mathbb{C}^n$ by

$$(H_f)_x(v, w) := \bar{v}^t f(x) w.$$

We obtain H_f compatible with \lrcorner ,
hence descends to a metric
on $\tilde{S} \times \mathbb{C}^n$. \square

Step 2: Relation between $\tilde{\nabla}_H$ and df .

Recall $D = \begin{matrix} \nabla^H \\ \uparrow \\ \tilde{\nabla}_H \end{matrix} + \begin{matrix} \tilde{\nabla}_H \\ \uparrow \\ \text{unitary self-adjoint} \end{matrix}$
 $d = D = \begin{matrix} \nabla^H \\ \uparrow \\ \tilde{\nabla}_H \end{matrix} + \begin{matrix} \tilde{\nabla}_H \\ \uparrow \\ \text{unitary self-adjoint} \end{matrix}$.

Claim: $\tilde{\nabla}_H = -\frac{1}{2} f^{-1} df$.

If: For two sections s, t of $\tilde{X} \times \mathbb{C}^n$,
by defn, $\tilde{H}_x(s, t) = \bar{s}^t f_H(x) t$.

We obtain.

$$(i) d(\tilde{H}(s, t)) = \tilde{H}(\nabla^H s, t) + \tilde{H}(s, \nabla^H t)$$

$$(ii) d(\tilde{H}(s, t)) = d(\bar{s}^t f_H t)$$

$$= d\bar{s}^t \cdot f_H \cdot t + \bar{s}^t d f_H t + \bar{s}^t f_H dt$$

Let s, t be constant sections (flat wrt d).

- $d(\hat{H}(s, t)) = \bar{s}^t \underline{\underline{df_H}} t.$
 - $d(\hat{H}(s, t)) = \hat{H}(-\bar{s}^t s, t) + \hat{H}(s, -\bar{s}^t t)$
 $= -2\bar{s}^t \underline{\underline{f_H}} \bar{s}^t t.$

$$\text{So } df_H = -2f_H \hat{\psi}_H$$

$$\Rightarrow \hat{\psi}_H = -\frac{1}{2} f_H^{-1} df_H.$$

Step 3: Claim: $\left\| \frac{\nabla f_H}{\|\cdot\|_{E(H)}} \right\|_{L^2}^2 = \frac{1}{4} \|df_H\|_{L^2}^2$

(They have the same critical pts.)

$$\begin{aligned}
 \text{Pf: } \langle df, df \rangle_{d\text{vol}_g} &= g_N(df \wedge *df) \\
 &= \text{tr}(f^{-1}df \wedge *f^{-1}df) \\
 &= 4 \text{tr}(\overset{\curvearrowleft}{\omega_H} \wedge \overset{\curvearrowright}{*}\omega_H).
 \end{aligned}$$

Conclusion: A harmonic metric on $(E, \tilde{g})^{\text{flat}}$ is equivalent to

a f -equiv harmonic map

$$f: X \rightarrow \frac{GL(n, \mathbb{C})}{U(n)}.$$

From now on, we can always associate to
a harmonic Higgs bundle (E, ϕ, H)

a ρ -equiv harmonic map $f: \tilde{X} \rightarrow \frac{GL(n, \mathbb{C})}{U(n)}$.

(Here $\rho = H\bar{d}(D)$, $D = T^H + \underbrace{\phi + \phi^*}_{\in H}$)

We write the data of f in terms of (E, ϕ, H) .

- Energy density

$$e_{\text{eff}} = \frac{1}{2} \|df\|_{g_0, g_N}^2 = 2 \operatorname{tr}(\tilde{\omega}_H \wedge * \tilde{\omega}_H) / \text{dvol } g_0.$$

$$= 2 \operatorname{tr}((\phi + \phi^*) \wedge *_{\mathbb{C}} (\phi + \phi^*)) / \text{dvol } g_0.$$

\mathbb{C} -linear.

$$*_{\mathbb{C}} dz = -i dz, \quad *_{\mathbb{C}} d\bar{z} = i d\bar{z}.$$

$$= 2 \operatorname{tr}((\phi + \phi^*) \wedge (-i\phi + i\phi^*)) / \text{dvol } g_0.$$

$$= 4i \operatorname{tr}(\phi \wedge \phi^*) / \text{dvol } g_0.$$

- Energy $E(f) = \int_X e(f) \text{dvol } g_0$

$$= 4i \int_X \operatorname{tr}(\phi \wedge \phi^*)$$

$$= 4 \int_X \operatorname{tr}(\phi \wedge *_h \phi)$$

$$= 4 \|\phi\|_{L^2}^2.$$

- The pullback metric

$$\begin{aligned}
 f^*g_N &= g_N(df \otimes df) = \text{tr}(f^{-1}df \otimes f^{-1}df) \\
 &= 4 \text{tr}(\tilde{\omega}_H \otimes \tilde{\omega}_H) \\
 &= 4 \text{tr}((\phi + \phi^*) \otimes (\phi + \phi^*)) \\
 &= 4 \text{tr}(\phi^2) + \underbrace{4 \text{tr}(\phi \phi^*)}_{(1,1)} + 4 \overline{\text{tr}(\phi^2)}_{(0,2)} + 4 \text{tr}(\phi^* \phi)
 \end{aligned}$$

- The Hopf differential

$$\text{Hopf}(f) := (f^*g_N)^{2,0} = 4 \text{tr}(\phi^2) \in H^0(X, K^2)$$

Note that it appears in the Hitchin fibration.

Hopf differential vanishes \Leftrightarrow f is conformal.

- One can express the sectional curvature of tangent plane

$$K_o^N = - \frac{\|[\phi, \phi^*]\|_{h, g_0}^2}{\|\phi\|_{h, g_0}^4 - |\text{tr}(\phi^2)|_{g_0}^2} \leq 0.$$

Moreover, $K_{f^*g_N} \leq K_o^N$.

§3. NAH.

Defn. The moduli space of flat connections:

$$M_{\text{deRham}}(n) := \{(E, D) \mid \begin{array}{l} E - \text{rk } n \text{ v.b} \\ D - \text{reductive flat connection} \end{array}\}$$

$$M_{\text{deRham}}^S(n) := \{(E, D) \mid \begin{array}{l} E - - \\ D - \text{irreducible flat conn} \end{array}\}$$

Defn. Call a rep $\rho: \pi_1(S) \rightarrow GL(n, \mathbb{C})$

semisimple / irreducible

if their associated rep on \mathbb{C}^n

are completely reducible / irreducible.

Defn. The moduli space of representations:

$$M_{\text{Betti}}(GL(n, \mathbb{C})) := \text{Hom}^{\text{semisimple}}(\pi_1(S), GL(n, \mathbb{C})) / GL(n, \mathbb{C})$$

$$M_{\text{Betti}}^S(GL(n, \mathbb{C})) := \text{Hom}^{\text{irreducible}}(\pi_1(S), GL(n, \mathbb{C})) / GL(n, \mathbb{C}).$$

semisimple \Rightarrow Hausdorff

irreducible \Rightarrow smooth.

Observe : Under the holonomy map, we have

irreducible flat connection \iff irreducible rep
reductive $\dashv \dashv \iff$ semisimple rep.

Follows from:

If P has a subrep space $\mathbb{C}^k \subset \mathbb{C}^n$,
 $\mathcal{X}_P \mathbb{C}^k$ is a D -inv subbundle
of $\mathcal{X}_P \mathbb{C}^n$.

If F is a D -inv subbundle of E ,
then $F_{x_0} \subset E_{x_0}$ is a subrep space
of $\rho = \text{Hol}(D) : \pi_1(X, x_0) \rightarrow \underline{\text{GL}(E_{x_0})}$.

Non-abelian Hodge correspondence

The following moduli spaces are homeo to each other
which restrict to diffeo on smooth part.

- (1) M_{Betti}
- (2) M_{deRham}
- (3) M_{Higgs}

$$\begin{aligned}
 (4) \quad M_{\text{Harm}} &\iff \{(E, D, H)\} / \text{gauge.} \\
 &\iff \{(E, \bar{\partial}_E, \Phi, H)\} / \text{gauge.} \\
 &\iff \{(P, f) \mid P: \pi_1 S \rightarrow \text{GL}(n, \mathbb{C})\} \\
 &\qquad \qquad \qquad f: X \rightarrow \text{GL}(n, \mathbb{C}) \\
 &\qquad \qquad \qquad \text{P-equiv, harmonic} \Big/ \text{GL}(n, \mathbb{C}) \\
 &\qquad \qquad \qquad g \cdot (P, f) = (g \cdot P, g \cdot f).
 \end{aligned}$$

Rmk: One can consider the non-abelian Hodge correspondence for real reductive Lie grps.

e.g. $G = \mathrm{PGL}(n, \mathbb{C}/\mathbb{R})$, $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SO}(n, \mathbb{C})$
 $\mathrm{SL}(n, \mathbb{C}/\mathbb{R})$, $\mathrm{U}(p, q)$, ...

When $G = \mathrm{SL}(n, \mathbb{R})$,
 recall that the Hitchin section has holonomy
 in $\mathrm{SL}(n, \mathbb{R})$.

Because $\dim_{\mathbb{R}} M_{\text{Betti}}(\text{SL}(n, \mathbb{R})) = \dim_{\mathbb{R}} B$,

$$2g \cdot \frac{(n^2-1)}{\text{generator}} - \frac{(n^2-1)}{\text{relation}} - \frac{(n^2-1)}{\text{conjugation}} = (2g-2)(n^2-1).$$

The image of the section S forms an open and closed nonempty subspace of $M_{\text{Higgs}}(\text{SL}(n, \mathbb{R}))$.

$\Rightarrow \text{Im}(S)$ is a connected component of $M_{\text{Higgs}}(\text{SL}(n, \mathbb{R}))$.

Such component is called Hitchin component, Hit_n .

Recall $s: \mathcal{B} \rightarrow M_{\text{Higgs}}(\text{SL}(n, \mathbb{C}))$

$$(q_2, \dots, q_n) \mapsto [E = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \dots \oplus K^{\frac{1-n}{2}},$$

$$\phi = \begin{pmatrix} 0 & q_2 & q_3 & \cdots & q_n \\ r_1 & 0 & q_2 & \ddots & \vdots \\ r_2 & r_3 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & q_3 \\ & & & \ddots & q_2 \\ & & & & r_{n-1} \end{pmatrix}].$$

$$r_i = \frac{i(n-i)}{2}.$$

(E, ϕ) is sym w.r.t $Q = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \end{pmatrix}$

A rep in Hit_n is called Hitchin rep.

Thm (Labourie) Hitchin reps are discrete, faithful and Anosov.

The orbit map $T_p: \pi_1 S \rightarrow \frac{\text{GL}(n, \mathbb{C})}{\text{U}(n)}$

$\delta \mapsto \delta x_0$
 is quasi-isometric.
 When $n=2$, Hitchin component is Teichmüller space
 \cap
 $M_{\text{Betti}}(\text{SL}(2, \mathbb{R}))$

Such rep: $\pi_1(S) \rightarrow \text{SL}(2, \mathbb{R})$ are called Fuchsian.
 They all arise from uniformizing
 hyp str on S .

Composing with the unique irred rep
 $l: \text{SL}(2, \mathbb{R}) \rightarrow \text{SL}(n, \mathbb{R})$,
 one can embed Fuchsian reps into $M_{\text{Betti}}(\text{SL}(n, \mathbb{R}))$,
 called n -Fuchsian reps.

Correspondingly, they are $S(0, 0, \dots, 0)$.

In particular, $S(0, 0, \dots, 0)$,
 solving Hitchin eqn
 is the same as
 uniformization of the R.S X .

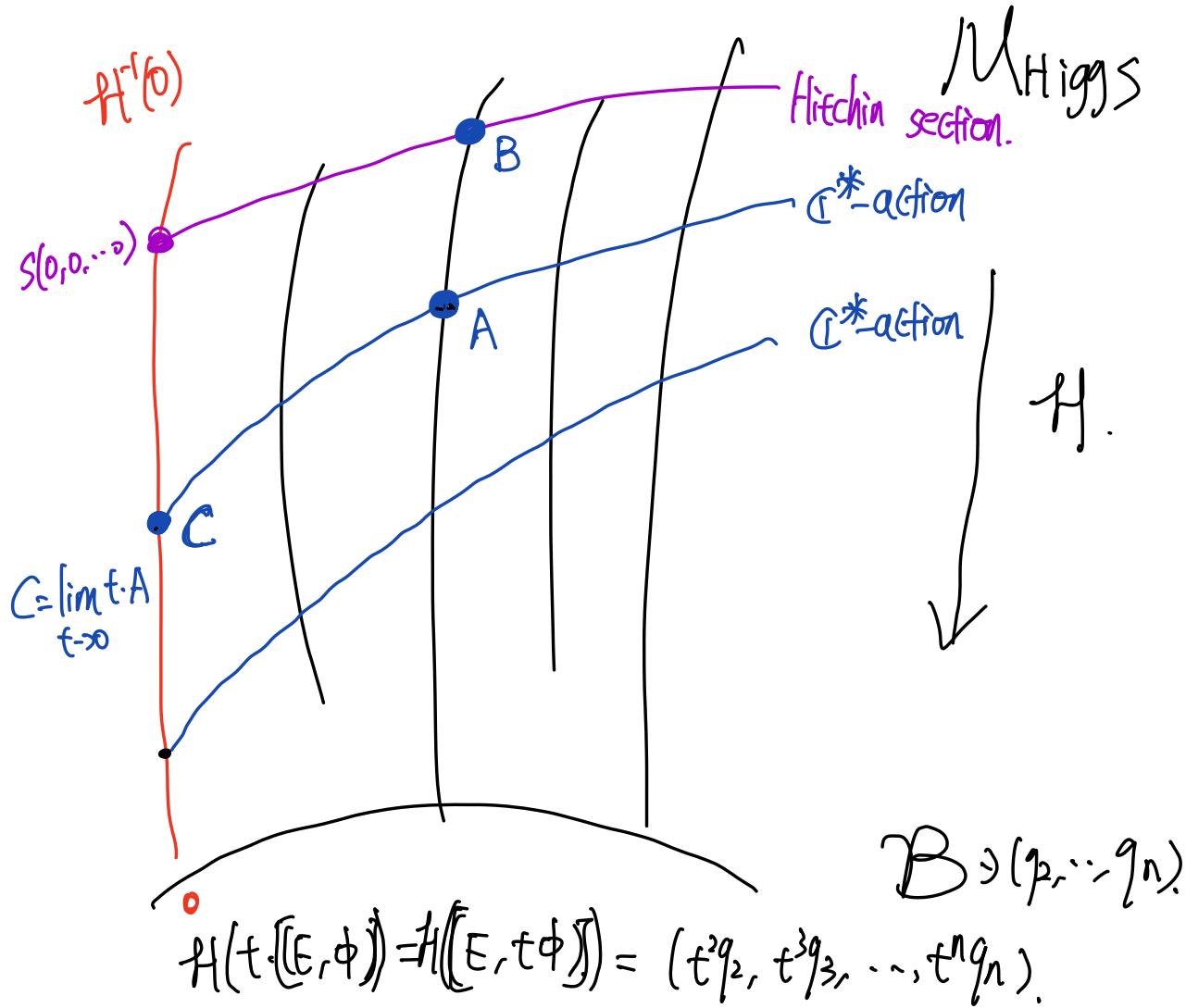
§4. State several conjecture by Song. Dai - L.

On $M^{\text{Higgs}}(SL(n, \mathbb{C}))$, consider

- \mathbb{C}^* -action
- Hitchin fibration $H: M \rightarrow \mathcal{B}$.

Define $H^{-1}(0)$ the nilpotent cone.

$(E, \phi) \in H^{-1}(0) \Leftrightarrow \phi$ is nilpotent.



$$f(E, \phi) = (q_1, \dots, q_n).$$

Given a pf $A \in M$, one can determine B, C .

Roughly, ask

energy density " $e_C \leq e_A \leq e_B$ " ?

Energy " $E_C \leq E_A \leq E_B$ " ?